

**Dual superconductivity,
monopole condensation and confining string
in low-energy Yang-Mills theory. Part I.**

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Abstract

We show that the QCD vacuum (without dynamical quarks) is a dual superconductor at least in the low-energy region in the sense that monopole condensation does really occur. In fact, we derive the dual Ginzburg-Landau theory (i.e., dual Abelian Higgs model) directly from the SU(2) Yang-Mills theory by adopting the maximal Abelian gauge. The dual superconductor can be on the border between type II and type I, excluding the London limit. The masses of the dual Abelian gauge field is expressed by the Yang-Mills gauge coupling constant and the mass of the off-diagonal gluon of the original Yang-Mills theory. Moreover, we can rewrite the Yang-Mills theory into an theory written in terms of the Abelian magnetic monopole alone at least in the low-energy region. Magnetic monopole condensation originates in the non-zero mass of off-diagonal gluons. Finally, we derive the confining string theory describing the low-energy Gluodynamics. Then the area law of the large Wilson loop is an immediate consequence of these constructions. Three low-energy effective theories give the same string tension.

Key words: quark confinement, magnetic monopole, QCD, confining string, monopole condensation, dual superconductivity,

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Contents

1	Introduction	1
2	Dual superconductivity in low-energy Gluodynamics	6
2.1	Conventions	6
2.2	Step 1: Non-Abelian Stokes theorem for the Wilson loop	7
2.3	Step 2: Cumulant expansion	9
2.4	Step 3: Maximal Abelian gauge fixing	10
2.5	Step 4: Dynamical mass generation for off-diagonal components . . .	12
2.6	Step 5: Low-energy effective theory for diagonal fields	12
2.6.1	Feynman rules	15
2.6.2	APEGT	17
2.7	Step 6: Dynamical generation of the kinetic term of $B_{\mu\nu}$	18
2.8	Step 7: Dual transformations	22
2.9	Step 8: Recovery of hypergauge symmetry and gauge fixing	24
2.10	Step 9: Change of variables (path-integral duality transformation) . .	26
2.11	Step 10: Dual Ginzburg-Landau theory in the London limit	28
3	Final step: Dual Ginzburg-Landau theory of the general type	30
3.1	Dual gauge theory	30
3.2	Low-energy effective theory of dual Abelian Higgs model	31
3.3	Comparison with the previous work	35
3.4	How to determine the parameters ρ, σ, α	36
4	Estimation of neglected higher-order terms	38
4.1	Higher-order cumulants and large N suppression	39
4.2	Higher-order terms of low-energy or large mass expansion and the de- coupling theorem	40
5	Magnetic monopole condensation and area law	41
5.1	Monopole action and monopole condensation	41
5.2	Area law of the Wilson loop	44
5.3	Type of dual superconductivity	46
6	Confining string theory and string tension	47
7	Parameter fitting for numerical estimation	52
7.1	Dual Ginzburg-Landau theory	52
7.2	Monopole action	52
7.3	Confining string	53
8	Conclusion and discussion	53

A	Useful formulae	55
A.1	Structure constants	55
A.2	Differential forms	55
A.3	Integration formula by dimensional regularization	56
A.4	Gamma function	56
B	Derivation of a version of non-Abelian Stokes theorem	56
C	Calculation of the vacuum polarization for tensor fields	57
D	Manifest covariant quantization of the second rank antisymmetric tensor gauge field	61
D.1	Second-rank antisymmetric tensor gauge theory	61
D.2	Inclusion of mass term	63
E	Renormalization of dual Abelian Higgs model	66
E.1	Goldstone bosons remain massless in higher orders	66
E.2	Vacuum polarization	67
F	Calculation of the Wilson loop	71
G	Derivative expansion of the string	73

1 Introduction

The main aim of this paper is to discuss how the dual superconductor picture for explaining quark confinement is derived directly from Quantum Chromodynamics (QCD). It is believed that quark and gluon are confined into the inside of hadrons by the strong interaction described by QCD and that they can not be observed in isolation. Quark confinement can be understood based on an idea of the electro-magnetic duality of ordinary superconductivity, i.e., the dual superconductor picture[1]. In this picture, the color electric flux can be excluded from QCD vacuum as a dual superconductor (the dual Meissner effect), just as the magnetic field can not penetrate into the superconductor (the so-called Meissner effect). In this context, the dual is used as implying electro-magnetic duality. However, QCD is a non-Abelian gauge theory and its gluonic part is described by the Yang-Mills theory, whereas the usual superconductivity is described by the Abelian gauge theory. Therefore we must make clear the precise meaning of dual superconductivity in QCD. It is possible to assume that the diagonal component of color electric field can be identified with the dual of magnetic field in the ordinary superconductivity. If so, we must answer what is the role played by the off-diagonal component of gluons? Finally, we must show that QCD vacuum has a tendency to exclude the (diagonal component of) color electric field. If a pair of heavy quark and anti-quark is immersed in the QCD vacuum, the color electric flux connecting a pair is squeezed into the tube-like region, leading to

the formation of QCD (gluon) string between a quark and an anti-quark. As a result, the interquark potential $V(R)$ is proportional to the interquark distance R . To separate a quark from an anti-quark, we need infinite energy. In this sense, quark confinement is achieved.

The dual superconductor picture of QCD vacuum is based on the following assumptions:

1. Existence of magnetic monopole: QCD has magnetic monopole.
2. Monopole condensation: Magnetic monopole is condensed in QCD.
3. Infrared Abelian dominance: Charged gluon (i.e., off-diagonal gluon) can be neglected at least in the low-energy region of QCD.
4. Absence of quantum effect: Classical configuration, e.g., the magnetic monopole, is dominant. Hence quantum effect can be neglected.

Each of these statements should be explained based on QCD to really confirm the dual superconductor picture of QCD vacuum. As a first step, a prescription of extracting the field configuration corresponding to magnetic monopole was proposed by 't Hooft [2]. This idea is called the Abelian projection. Immediately after the proposal of this idea, infrared Abelian dominance was suggested by Ezawa and Iwazaki [3]. However, recent simulations revealed that the Abelian dominance is not necessarily realized in all the Abelian gauges, see the review [4]. To author's knowledge, the best covariant gauge fixing condition realizing Abelian dominance is given by the maximal Abelian (MA) gauge proposed by Kronfeld et al. [5]. In fact, the infrared Abelian dominance is first confirmed in MA gauge based on Monte Carlo simulations by Suzuki and Yotsuyanagi [6]. Subsequent simulations have also confirmed the monopole dominance in low-energy QCD [7]. An analytical derivation of the dual Ginzburg-Landau (DGL) theory was tried by Suzuki [8] by way of Zwanziger formulation [9] by neglecting the off-diagonal components of gluon fields by virtue of the Abelian dominance and by assuming condensation of magnetic monopoles. However, the DGL theory is not yet derived directly from the underlying theory, i.e. QCD, since the Abelian dominance and monopole condensation themselves must be derived in the same framework of the theory. According to the recent Monte Carlo simulations [10], the presumed dual Ginzburg-Landau theory is on the border of type II and type I.

In this paper we discuss how to derive the DGL theory as a low-energy effective theory (LEET) of QCD. The LEET is not unique and we can derive various LEET's. Of course, they should be equivalent to each other, if they are to be derived directly from QCD. Even if a specific LEET of QCD is assumed, the parameters included in the LEET can be adjusted so as to meet the data of experiments or Monte Carlo simulation on a lattice. In the first paper [11] of a series of papers [12, 13, 14, 15, 16] on the quark confinement in Yang-Mills theory in the Abelian gauge, the author has given a scenario of deriving the dual superconductivity in low-energy region of QCD and demonstrated that the DGL theory, i.e., the dual Abelian Higgs (DAH) theory in the London limit can be derived from QCD, provided that the condensation of

magnetic monopole takes place. In this scenario, the mass m_b of the dual gluon (dual Abelian gauge field) is given directly by the non-zero condensate of the magnetic monopole current k_μ , i.e., $\langle k_\mu k_\mu \rangle \neq 0$. Since the DGL theory is a LEET written in term of only the Abelian (diagonal) component extracted from the original non-Abelian field, the treatment of the off-diagonal component is crucial for deriving the LEET and also for giving reasonable interpretation of the result. In the previous paper [11], all the off-diagonal components are integrated out to write down the LEET in terms of the diagonal components alone. The resulting LEET is an Abelian gauge theory preserving a characteristic feature of non-Abelian gauge theory in the sense that the β function for the coupling constant has the same form as the original Yang-Mills theory, exhibiting the asymptotic freedom [17, 11]. For this procedure to be meaningful, the mass M_A of the off-diagonal components must be heavier than the mass of the diagonal components. This procedure can be justified based on the Wilsonian renormalization group or the decoupling theorem [18]. As a result, the LEET is valid in the energy region below M_A .

The aim of this paper is to show that the dual superconductivity can be derived at least in the low-energy region of Gluodynamics (i.e., the gluonic sector of QCD). The most important ingredient in deriving the dual superconductivity is the existence of non-zero mass of the off-diagonal gluons. Recently, it has been shown [19, 20] that the off-diagonal gluons and off-diagonal ghosts (anti-ghosts) acquire non-zero masses in Yang-Mills theory in the *modified* MA gauge [20]. This result strongly supports the Abelian dominance in low-energy Gluodynamics. The modified MA gauge was already proposed by the author from a different viewpoint in the paper [12]. A remarkable difference of MA gauge from the usual Lorentz type gauge lies in the fact that MA gauge is a nonlinear gauge. In order to preserve the renormalizability of Yang-Mills theory in the MA gauge, it is indispensable to introduce quartic ghost interaction term as a piece of gauge fixing term [21]. The modified MA gauge fixes the strength of the quartic ghost interaction by imposing the symmetry, i.e., orthosymplectic group $OSp(4|2)$. The implications of the $OSp(4|2)$ symmetry have been discussed in the previous papers [12, 13, 14]. The attractive quartic ghost interaction causes the ghost–anti-ghost condensation. Consequently, the off-diagonal gluons become massive,¹ whereas the diagonal gluons remain massless in this gauge. Since the classical Yang-Mills theory is a scale invariant theory, the mass scale must be generated due to quantum effect. In this paper, we show that *the monopole condensation does really occur due to existence of non-zero mass of off-diagonal gluons. The off-diagonal gluon mass also provides the mass of dual gauge field in the DGL theory.*

It should be remarked that in the conventional approaches the off-diagonal components are completely neglected from the beginning in deriving the effective Abelian gauge theory by virtue of the Abelian dominance. As a result, the conventional approach can not predict the physical quantities without some fitting of the parameters introduced by hand. The purpose of this paper is to bridge between the perturbative

¹The off-diagonal gluon mass M_A in the MA gauge has been calculated on a lattice by Amemiya and Suganuma [23], $M_A = 1.2\text{GeV}$ for $SU(2)$.

QCD in the high-energy region and the low-energy effective Abelian gauge theory. Consequently, the undetermined parameters in the low-energy effective theory can in principle be expressed by the parameters of the original Yang-Mills theory, i.e., the gauge coupling constant g and the renormalization group (RG) invariant scale Λ_{QCD} .

In this paper we pay attention to the vacuum expectation value (VEV) of the Wilson loop operator $W_C[\mathcal{A}]$, which is written in the framework of the functional integration as

$$\langle W_C[\mathcal{A}] \rangle_{YM} = Z_{YM}^{-1} \int d\mu \exp \left\{ i \int d^4x \mathcal{L}_{YM}^{tot} \right\} W_C[\mathcal{A}], \quad (1.1)$$

where \mathcal{L}_{YM}^{tot} is the total Yang-Mills Lagrangian, i.e., Yang-Mills Lagrangian \mathcal{L}_{YM} plus the gauge fixing (GF) term \mathcal{L}_{GF+FP} including the Faddeev-Popov (FP) ghost term in the modified MA gauge. Our derivation is based on the Becchi-Rouet-Stora-Tyutin [22](BRST) formulation of the gauge theory. Hence, $d\mu$ is the integration measure which is BRST invariant, $d\mu_{YM} := \mathcal{D}\mathcal{A}_\mu \mathcal{D}B \mathcal{D}C \mathcal{D}\bar{C}$, where B is the Nakanishi-Lautrap (NL) Lagrangian multiplier field. We regard the Wilson loop as a special choice of the source term in general Yang-Mills theory. The Wilson loop is used as a probe to see the QCD vacuum. The shape of the Wilson loop is arbitrary at this stage.

We seek other theories (with the corresponding source term) which is equivalent to the original Yang-Mills theory at least in the low-energy region in the sense that it gives the same VEV of the Wilson loop operator as the original Yang-Mills theory for *large loop* C . From this viewpoint, we obtain three LEET's, i.e., DGL theory (DAH theory), magnetic monopole theory and confining string theory. For this derivation to be successful, the existence of quantum corrections coming from off-diagonal components of gluons and ghost (and anti-ghost) is indispensable. Without quantum corrections, we can not derive magnetic monopole condensation. This is plausible, since it is the quantum correction that can introduce the scale into the gauge theory which is scale invariant at the classical level.

This paper is organized as follows. In section 2, we give a strategy (steps) of deriving the LEET of Gluodynamics. In fact, we demonstrate that at least the London limit of the DGL theory (i.e., DAH model) of type II can be obtained as a very special limit of the resulting LEET of Gluodynamics. However, our method is able to treat more general situation, not restricted to the London limit. Rather, our result suggests that the DGL theory can not be in the London limit.

In section 3, a more general case of the DGL theory is discussed. It is shown that the LEET of the supposed DGL theory agrees with the LEET derived from the Yang-Mills theory according to the above steps, in the energy region less than the mass m_H of the dual Higgs mass. This way of showing equivalence is a little bit indirect. The reason is as follows. We know that the DAH model or the DGL theory is a renormalizable theory (within perturbation theory in the magnetic coupling constant $g_m = 4\pi/g$) and hence is a meaningful theory at arbitrary energy scale. On the other hand, we know that the high energy region of Gluodynamics is correctly described by the non-Abelian Yang-Mills theory with asymptotic freedom. Therefore, the DGL theory is at best meaningful only in the low-energy region in this context, although it

is a renormalizable theory. In this sense, we must be careful in saying that the DGL theory is regarded as a LEET of Gluodynamics.

In section 4, we try to estimate the neglected terms in the derivation of the LEET based on the large N argument and the decoupling theorem. It is shown that in the large N expansion the higher-order terms are suppressed by N^{-2} compared to the leading order term. Thus the LEET obtained above is considered as the leading-order result of the large N expansion.

The LEET of Gluodynamics is not unique. In fact, we can obtain various LEET's which are equivalent to each other. Other LEET's can be more convenient for the purpose of calculating some kinds of physical quantities.

In section 5, we derive the magnetic monopole action as another LEET. The magnetic monopole action $S_{MP}[k]$ is written entirely in terms of the magnetic monopole current k_μ .

$$\langle W_C[\mathcal{A}] \rangle_{YM} = Z_{MP}^{-1} \int \mathcal{D}k_\mu \exp \left\{ i \int d^4x (\mathcal{L}_{MP}[k] + k_\mu \Xi^\mu) \right\}, \quad (1.2)$$

by introducing the four-dimensional solid angle Ξ_μ . We show that the VEV of the Wilson loop exhibits area law decay for large loop C ,

$$\langle W_C[\mathcal{A}] \rangle_{YM} \cong \exp \{ -\sigma_{st} A(C) \}, \quad (1.3)$$

where $A(C)$ is the area of the minimal surface S bounded by C . The string tension σ_{st} is explicitly obtained and it agrees with that predicted by the DGL theory.. Then we show that the magnetic monopole condensation really occurs in the low-energy region in the sense that the current-current correlation at the same space-time point, $\langle k_\mu k_\mu \rangle$, has non-vanishing expectation value,

$$\langle k_\mu k^\mu \rangle_{YM} := Z_{MP}^{-1} \int \mathcal{D}k_\mu \exp \left\{ i \int d^4x \mathcal{L}_{MP}[k] \right\} k_\mu k^\mu \neq 0. \quad (1.4)$$

The result is consistent with that of the monopole action on a lattice[24].

In section 6, we derive a string theory which is equivalent to the LEET's derived above,

$$\langle W_C[\mathcal{A}] \rangle_{YM} = Z_{cs}^{-1} \int \mathcal{D}x_\mu(\sigma) \exp \{ i S_{cs}[x] \}. \quad (1.5)$$

The action $S_{cs}[x]$ of the confining string is equal to the Nambu-Goto action for the world sheet S with the Wilson loop C as the boundary. The string tension σ_{cs} in the Nambu-Goto action is the same as σ_{st} evaluated by the monopole action derived in section 5. The Wilson loop exhibits area law with the string tension σ_{st} . Therefore, the obtained string theory is regarded as a low-energy limit of the so-called confining string theory proposed by Polyakov [25].

In section 7, we discuss what values of the parameters in the LEET's should be chosen to reproduce the numerical results.

The final section 8 is devoted to summarizing the result obtained in this paper and discussing the future directions of our investigations. The details of calculations are collected in Appendices, together with useful formulae.

2 Dual superconductivity in low-energy Gluodynamics

2.1 Conventions

The gauge potential \mathcal{A}_μ is written as

$$\mathcal{A}_\mu(x) := \mathcal{A}_\mu^A(x) T^A \quad (A = 1, \dots, N^2 - 1), \quad (2.1)$$

where the generators $T^A (A = 1, \dots, N^2 - 1)$ of the Lie algebra \mathcal{G} of the gauge group $G = SU(N)$ are taken to be Hermitian satisfying $[T^A, T^B] = if^{ABC} T^C$ and normalized as $\text{tr}(T^A T^B) = \frac{1}{2} \delta^{AB}$. For a closed loop C , we define the Wilson loop operator $W(C)$ by

$$W(C) = \text{tr} \left[P \exp \left\{ ig \oint_C dx^\mu \mathcal{A}_\mu(x) \right\} \right] / \text{tr}(1), \quad (2.2)$$

where P denotes the path-ordered product and g is the Yang-Mills coupling constant.

We begin with the vacuum expectation value (VEV) of the Wilson loop operator $W(C)$ in the Yang-Mills theory with a gauge group $SU(N)$ defined by the functional integral,

$$\langle W(C) \rangle_{YM} := Z_{YM}^{-1} \int d\mu_{YM} \exp(i S_{YM}^{tot}) W(C), \quad (2.3)$$

where Z_{YM} is a normalization factor (or a partition function) to guarantee $\langle 1 \rangle_{YM} \equiv 1$ and S_{YM}^{tot} is the total action obtained by adding the gauge fixing (GF) and Faddeev-Popov (FP) ghost term S_{GF+FP} to the Yang-Mills action S_{YM} ,

$$S_{YM}^{tot} = S_{YM} + S_{GF+FP}. \quad (2.4)$$

The Yang-Mills action is of the usual form,

$$S_{YM} = \int d^4x \frac{-1}{4} \mathcal{F}_{\mu\nu}^A(x) \mathcal{F}^{\mu\nu A}(x), \quad (2.5)$$

where $\mathcal{F}_{\mu\nu}(x)$ is the field strength defined by

$$\mathcal{F}_{\mu\nu}(x) := \mathcal{F}_{\mu\nu}^A(x) T^A := \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) - ig [\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)]. \quad (2.6)$$

The GF+FP action S_{GF+FP} is specified later (see section 2.4). Finally, $d\mu_{YM}$ is the integration measure,

$$d\mu_{YM} := \mathcal{D}\mathcal{A}_\mu^A \mathcal{D}B^A \mathcal{D}\mathcal{C}^A \mathcal{D}\bar{\mathcal{C}}^A, \quad (2.7)$$

which is invariant under the Becchi-Rouet-Stora-Tyutin (BRST) transformation,

$$\begin{aligned} \delta_B \mathcal{A}_\mu(x) &= \mathcal{D}_\mu [\mathcal{A}] \mathcal{C}(x) := \partial_\mu \mathcal{C}(x) - ig [\mathcal{A}_\mu(x), \mathcal{C}(x)], \\ \delta_B \mathcal{C}(x) &= i \frac{g}{2} [\mathcal{C}(x), \mathcal{C}(x)], \\ \delta_B \bar{\mathcal{C}}(x) &= i B(x), \\ \delta_B B(x) &= 0, \end{aligned} \quad (2.8)$$

where B is the Nakanishi-Lautrap (NL) field and \mathcal{C} ($\bar{\mathcal{C}}$) is the ghost (anti-ghost) field.

2.2 Step 1: Non-Abelian Stokes theorem for the Wilson loop

We make use of a version of the non-Abelian Stokes theorem (NAST) [26, 14, 27] to rewrite the Wilson loop operator in terms of the diagonal components. This version of NAST was first obtained by Diakonov and Petrov for $SU(2)$ [26]. It is possible to generalize their result to $SU(N)$ ($N \geq 3$).

Theorem:[27] *For a closed loop C , we define the non-Abelian Wilson loop operator by*

$$W_C[\mathcal{A}] = \text{tr} \left\{ P \exp \left[ig \oint_C dx^\mu \mathcal{A}_\mu(x) \right] \right\} / \text{tr}(1), \quad (2.9)$$

where P is the path-ordered product. Then it is rewritten as

$$W_C[\mathcal{A}] = \int d\mu_C(\xi) \exp \left[ig \oint_C dx^\mu a_\mu^\xi(x) \right] = \int d\mu_C(\xi) \exp \left[ig \int_S d\sigma^{\mu\nu} f_{\mu\nu}^\xi(x) \right], \quad (2.10)$$

where

$$a_\mu^\xi(x) := \langle \Lambda | \mathcal{A}_\mu^\xi(x) | \Lambda \rangle, \quad \mathcal{A}_\mu^\xi(x) = \xi^\dagger(x) \mathcal{A}_\mu(x) \xi(x) + \frac{i}{g} \xi^\dagger(x) \partial_\mu \xi(x), \quad (2.11)$$

and²

$$f_{\mu\nu}^\xi(x) := \partial_\mu a_\nu^\xi(x) - \partial_\nu a_\mu^\xi(x). \quad (2.12)$$

Here $|\Lambda\rangle$ is the highest-weight state of the representation defining the Wilson loop and the measure $d\mu_C(\xi)$ is the product measure along the loop C , $d\mu_C(\xi) = \prod_{x \in C} d\mu(\xi(x))$, where $d\mu(\xi)$ is the invariant Haar measure on G/\tilde{H} with the maximal stability group \tilde{H} . The maximal stability group \tilde{H} is the subgroup leaving the highest-weight state invariant (up to a phase factor), i.e.,

$$g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)}, \quad (2.13)$$

for $\xi \in G/\tilde{H}$ and $h \in \tilde{H}$. It depends on the group G and the representation in question.

For $G = SU(2)$, the \tilde{H} is given by the maximal torus subgroup $H = U(1)$ irrespective of the representation. Hence $G/\tilde{H} = SU(2)/U(1) = CP^1 = F_1$. For $G = SU(N)$ ($N \geq 3$), however, \tilde{H} does not necessarily agree with the maximal torus group $H = U(1)^{N-1}$ depending on the representation. For $G = SU(3)$, all the representations can be classified by the Dynkin index $[m, n]$. If $m = 0$ or $n = 0$, $\tilde{H} = U(2)$ and $G/\tilde{H} = CP^2$. On the other hand, when $m \neq 0$ and $n \neq 0$, $\tilde{H} = U(1)^2 = U(1) \times U(1)$ and $G/\tilde{H} = F_2$. Here CP^n is the complex projective space and F_n the flag space. This NAST is obtained by making use of the generalized coherent state. For details, see the reference[27].

² Note that $f_{\mu\nu}$ is not equal to the diagonal component of $\mathcal{F}_{\mu\nu}$.

For the fundamental representation, the expression (2.11) is greatly simplified as

$$\langle \Lambda | (\cdots) | \Lambda \rangle = 2\text{tr}[\mathcal{H}(\cdots)], \quad \mathcal{H} = \frac{1}{2} \text{diag} \left(\frac{N-1}{N}, \frac{-1}{N}, \dots, \frac{-1}{N} \right). \quad (2.14)$$

Therefore,

$$a_\mu = \frac{1}{2} \mathcal{A}_\mu^3 \quad \text{for } G = SU(2) \quad (2.15)$$

$$a_\mu = \frac{1}{3} \left[\mathcal{A}_\mu^3 + \frac{1}{\sqrt{3}} \mathcal{A}_\mu^8 \right] \quad \text{for } G = SU(3). \quad (2.16)$$

This implies that the non-Abelian Wilson loop can be expressed by the diagonal (Abelian) components. This is suggestive of the Abelian dominance in the expectation value of the Wilson loop.

The monopole dominance in the Wilson loop is also expected to hold as shown follows. We can rewrite $f_{\mu\nu}^\xi$ in the NAST as

$$f_{\mu\nu}^\xi(x) = \partial_\mu[n^A(x)\mathcal{A}_\nu^A(x)] - \partial_\nu[n^A(x)\mathcal{A}_\mu^A(x)] - \frac{1}{g}f^{ABC}n^A(x)\partial_\mu n^B(x)\partial_\nu n^C(x), \quad (2.17)$$

where

$$n^A(x)T^A = \xi(x)\mathcal{H}\xi^\dagger(x) = U(x)\mathcal{H}U^\dagger(x). \quad (2.18)$$

The $f_{\mu\nu}^\xi$ is invariant under the full G gauge transformation as well as the residual H gauge transformation. (Indeed, we can write a manifestly gauge invariant form, see ref.[28].) The $f_{\mu\nu}^\xi$ is a generalization of the 't Hooft-Polyakov tensor of the non-Abelian magnetic monopole, if we identify $n^A(x)$ with the unit vector of the elementary Higgs scalar field in the gauge-Higgs theory:

$$n^A(x) \leftrightarrow \hat{\phi}^A(x) := \phi^A(x)/|\phi(x)|. \quad (2.19)$$

This implies that $n^A(x)$ is identified with the composite scalar field and plays the same role as the scalar field in the gauge-Higgs model, even though QCD has no elementary scalar field. This fact could explain why the QCD vacuum can be dual superconductor due to magnetic monopole condensation.

By introducing the magnetic monopole current k by $k := \delta * f$ (see Appendix A), we have another expression,

$$W_C[\mathcal{A}] = \int d\mu_C(\xi) \exp \left[ig(\Xi, k^\xi) \right], \quad \Xi := \Delta^{-1} * d\Theta, \quad (2.20)$$

where Δ is the Laplacian $\Delta := d\delta + \delta d$ and T is a two-form determined by the surface element dS of the surface spanned by the Wilson loop C . The derivation is given in Appendix B. Hence, the Wilson loop can also be expressed by the magnetic monopole current k_μ . The above results hold irrespective of which gauge theory we consider.

In the case of $SU(2)$, the Wilson loop in an arbitrary representation is written in the form [26, 14],

$$W_C[\mathcal{A}] = \int d\mu_C(\xi) \exp \left[igJ \oint_C dx^\mu a_\mu^\xi(x) \right], \quad (2.21)$$

where a_μ^ξ is the *Abelian* gauge field (or the diagonal component of \mathcal{A}_μ^ξ) defined by

$$a_\mu^\xi(x) := \text{tr}\{\sigma_3[\xi(x)\mathcal{A}_\mu(x)\xi^\dagger(x) + ig^{-1}\xi(x)\partial_\mu\xi^\dagger(x)]\}, \quad (2.22)$$

for an element $\xi \in G/H = SU(2)/U(1) \cong S^2 \cong CP^1$. Here J is a character which distinguishes the different representation defining the Wilson loop, $J = \frac{1}{2}, 1, \frac{3}{2}, \dots$. Moreover, the use of the usual Stokes theorem leads to

$$W(C) = \int d\mu_C(\xi) \exp \left[igJ \int_{S_C} dS^{\mu\nu} f_{\mu\nu}^\xi(x) \right], \quad (2.23)$$

where $f_{\mu\nu}^\xi(x)$ is the *Abelian* field strength defined by $f_{\mu\nu}^\xi(x) := \partial_\mu a_\nu^\xi(x) - \partial_\nu a_\mu^\xi(x)$, and S_C is an arbitrary two-dimensional surface with the loop C as the boundary. Here it should be remarked that $f_{\mu\nu}^\xi(x)$ is invariant under the full $G = SU(2)$ gauge transformation as well as the residual $H = U(1)$ gauge transformation, since it has the same form as the usual 'tHooft–Polyakov tensor describing the magnetic monopole, see [14].

2.3 Step 2: Cumulant expansion

Now we specify the gauge theory in terms of which the VEV of the Wilson loop operator is evaluated. We consider the Yang-Mills theory with gauge fixing term. The gauge fixing is discussed in the next step.

By applying the cumulant expansion to the VEV of the Wilson loop,

$$\langle W(C) \rangle_{YM} = \int d\mu_C(\xi) \left\langle \exp \left[igJ \int_{S_C} dS^{\mu\nu} f_{\mu\nu}^\xi(x) \right] \right\rangle_{YM}, \quad (2.24)$$

the VEV of the exponential is replaced by the exponential of the connected expectation as

$$\langle W(C) \rangle_{YM} = \int d\mu_C(\xi) \exp \left[-\frac{g^2}{2} J^2 \int_{S_C} dS^{\mu\nu}(x) \int_{S_C} dS^{\rho\sigma}(y) \langle f_{\mu\nu}^\xi(x) f_{\rho\sigma}^\xi(y) \rangle_{YM} + \dots \right], \quad (2.25)$$

where we have used $\langle f_{\mu\nu}^\xi(x) \rangle_{YM} = 0$ and \dots denotes the higher-order cumulants.

The cumulant expansion is a well-known technique in statistical mechanics and quantum field theory. In what follows, we will neglect the higher-order cumulants in (2.25) as in the analysis of the stochastic vacuum model.³ The validity of this approximation, i.e, the truncation of the cumulant expansion, can be examined by Monte Carlo simulations on a lattice, as performed for the stochastic vacuum model, see [30]. In the framework of our approach, this approximation can be justified in the sense that the approximation is self-consistent within the APEGT derived below. See section 4.1 for more details.

³In the non-perturbative study of QCD, the cumulant expansion is extensively utilized by the stochastic vacuum model (SVM) [29] where the different version of the non-Abelian Stokes theorem is adopted. In the SVM, the approximation of neglecting higher order cumulants is called the bilocal approximation. The validity of bilocal approximation in SVM was confirmed by Monte Carlo simulation on a lattice [30, 31]. The author would like to thank Dmitri Antonov for this information.

2.4 Step 3: Maximal Abelian gauge fixing

First of all, we define the decomposition of the gauge potential into the diagonal and off-diagonal components,

$$\mathcal{A}_\mu(x) = \mathcal{A}_\mu^A(x)T^A = a_\mu^i(x)T^i + A_\mu^a(x)T^a, \quad (2.26)$$

where $T^i \in \mathcal{H}$ and $T^a \in \mathcal{G} - \mathcal{H}$ with \mathcal{H} being the Cartan subalgebra. As a gauge fixing condition for the off-diagonal component, we adopt the modified version of the maximal Abelian (MA) gauge proposed by the author [12],

$$S_{GF+FP} = \int d^4x i\delta_B \bar{\delta}_B \left[\frac{1}{2} A_\mu^a(x) A^{\mu a}(x) - \frac{\alpha}{2} i C^a(x) \bar{C}^a(x) \right], \quad (2.27)$$

where α corresponds to the gauge fixing parameter for the off-diagonal components, since the explicit calculation of the anti-BRST transformation $\bar{\delta}_B$ yields

$$S_{GF+FP} = - \int d^4x i\delta_B \left[\bar{C}^a \left\{ D_\mu[a] A^\mu + \frac{\alpha}{2} B \right\}^a - i \frac{\alpha}{2} g f^{abi} \bar{C}^a \bar{C}^b C^i - i \frac{\alpha}{4} g f^{abc} C^a \bar{C}^b \bar{C}^c \right]. \quad (2.28)$$

In order to see the effect of ghost self-interaction, we take

$$S_{GF+FP} = - \int d^4x i\delta_B \left[\bar{C}^a \left\{ D_\mu[a] A^\mu + \frac{\alpha}{2} B \right\}^a - i \frac{\zeta}{2} g f^{abi} \bar{C}^a \bar{C}^b C^i - i \frac{\zeta}{4} g f^{abc} C^a \bar{C}^b \bar{C}^c \right], \quad (2.29)$$

where we must put $\zeta = \alpha$ to recover Eq.(2.27). The most general form of the MA gauge was obtained by Hata and Niigata [32].

By performing the BRST transformation explicitly, we obtain

$$\begin{aligned} S_{GF+FP} = & \int d^4x \{ B^a D_\mu[a]^{ab} A^{\mu b} + \frac{\alpha}{2} B^a B^a \\ & + i \bar{C}^a D_\mu[a]^{ac} D^\mu[a]^{cb} C^b - i g^2 f^{adi} f^{cbi} \bar{C}^a C^b A^{\mu c} A_\mu^d \\ & + i \bar{C}^a D_\mu[a]^{ac} (g f^{cdb} A^{\mu d} C^b) + i \bar{C}^a g f^{abi} (D^\mu[a]^{bc} A_\mu^c) C^i \\ & + \frac{\zeta}{8} g^2 f^{abe} f^{cde} \bar{C}^a \bar{C}^b C^c C^d + \frac{\zeta}{4} g^2 f^{abc} f^{aid} \bar{C}^b \bar{C}^c C^i C^d + \frac{\zeta}{2} g f^{abc} i B^b C^a \bar{C}^c \\ & - \zeta g f^{abi} i B^a \bar{C}^b C^i + \frac{\zeta}{4} g^2 f^{abi} f^{cdi} \bar{C}^a \bar{C}^b C^c C^d \}. \end{aligned} \quad (2.30)$$

In particular, the $SU(2)$ case is greatly simplified as

$$\begin{aligned} S_{GF+FP} = & \int d^4x \{ B^a D_\mu[a]^{ab} A^{\mu b} + \frac{\alpha}{2} B^a B^a \\ & + i \bar{C}^a D_\mu[a]^{ac} D^\mu[a]^{cb} C^b - i g^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d \\ & + i \bar{C}^a g \epsilon^{ab} (D_\mu[a]^{bc} A_\mu^c) C^3 \\ & - \zeta g \epsilon^{ab} i B^a \bar{C}^b C^3 + \frac{\zeta}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d \}. \end{aligned} \quad (2.31)$$

Integrating out the NL field B^a leads to

$$\begin{aligned}
S_{GF+FP} = & \int d^4x \left\{ -\frac{1}{2\alpha} (D_\mu[a]^{ab} A^{\mu b})^2 + (1 - \zeta/\alpha) i \bar{C}^a g \epsilon^{ab} (D_\mu[a]^{bc} A_\mu^c) C^3 \right. \\
& + i \bar{C}^a D_\mu[a]^{ac} D^\mu[a]^{cb} C^b - i g^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d \\
& \left. + \frac{\zeta}{4} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d \right\}.
\end{aligned} \tag{2.32}$$

The advantages of the modified MA gauge (2.27) are as follows.

1. S_{GF+FP} is BRST invariant, i.e., $\delta_B S_{GF+FP} = 0$, due to nilpotency of the BRST transformation $\delta_B^2 = 0$.
2. S_{GF+FP} is anti-BRST invariant, i.e., $\bar{\delta}_B S_{GF+FP} = 0$, due to nilpotency of the anti-BRST transformation $\bar{\delta}_B^2 = 0$.
3. S_{GF+FP} is supersymmetric, i.e., invariant under the $OSp(4|2)$ rotation among the component fields in the supermultiplet $(\mathcal{A}_\mu, C, \bar{C})$ defined on the superspace $(x_\mu, \theta, \bar{\theta})$. The hidden supersymmetry causes the dimensional reduction in the sense of Parisi-Sourlas. Then the 4-dimensional GF+FP sector reduces to the 2-dimensional coset (G/H) nonlinear sigma model. See ref.[12] for more details.
4. S_{GF+FP} is invariant under the FP ghost conjugation,

$$C^A \rightarrow \pm \bar{C}^A, \quad \bar{C}^A \rightarrow \mp C^A, \quad B^A \rightarrow -\bar{B}^A, \quad \bar{B}^A \rightarrow -B^A, \quad (\mathcal{A}_\mu^A \rightarrow \mathcal{A}_\mu^A). \tag{2.33}$$

Therefore, we can treat C and \bar{C} on equal footing. In other words, the theory is totally symmetric under the exchange of C and \bar{C} .

5. The Yang-Mills theory in the modified MA gauge (with the total action $S_{YM} + S_{GF+FP}$) is renormalizable. The naive MA gauge

$$S_{GF+FP} = - \int d^4x i \delta_B \left[\bar{C}^a \left\{ D_\mu[a] A^\mu + \frac{\alpha}{2} B \right\}^a \right] \tag{2.34}$$

$$\begin{aligned}
= & \int d^4x \left\{ B^a D_\mu[a]^{ab} A^{\mu b} + \frac{\alpha}{2} B^a B^a \right. \\
& + i \bar{C}^a D_\mu[a]^{ac} D^\mu[a]^{cb} C^b - i g^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d \\
& \left. + i \bar{C}^a g \epsilon^{ab} (D_\mu[a]^{bc} A_\mu^c) C^3 \right\}
\end{aligned} \tag{2.35}$$

spoils the renormalizability, since radiative corrections induce (even for $\alpha = 0$) the four-ghost interaction,

$$z_{4c} g^2 \epsilon^{ab} \epsilon^{cd} \bar{C}^a \bar{C}^b C^c C^d, \quad z_{4c} = 4N \frac{g^2}{(4\pi)^2} \ln \frac{\mu}{\mu_0}, \tag{2.36}$$

owing to the existence of the vertex $-i g^2 \epsilon^{ad} \epsilon^{cb} \bar{C}^a C^b A^{\mu c} A_\mu^d$, see eq.(2.52) and Appendix B of the paper [11]. This is because the MA gauge is a nonlinear gauge. For the renormalizability of the Yang-Mills theory in the MA gauge, therefore, we need the four-ghost interaction from the beginning. In fact, the renormalizability of the Yang-Mills theory supplemented with the four-ghost interaction was proved to all orders in perturbation theory [21].

In order to completely fix the gauge degrees of freedom, we add the GF+FP term for the diagonal component a_μ^i to (2.28) or (2.29):

$$S_{GF+FP} = - \int d^4x i \delta_B \left[\bar{C}^i \left(\partial^\mu a_\mu^i + \frac{\beta}{2} B^i \right) \right], \quad (2.37)$$

where we have adopted the gauge fixing condition of the Lorentz type, $\partial^\mu a_\mu^i = 0$. The choice of the modified MA gauge is essential in deriving the off-diagonal gluon mass.

2.5 Step 4: Dynamical mass generation for off-diagonal components

It is shown that the four-ghost self-interaction is indispensable for the renormalizability of Yang-Mills theory in the MA gauge. Moreover, it has been shown [19, 20] that the attractive four-ghost interaction in the modified MA gauge causes the ghost–anti-ghost condensation and that this condensation provides masses for the off-diagonal gluons and off-diagonal ghosts and anti-ghosts. The massive off-diagonal fields do not propagate in the long distance. Therefore, we can neglect off-diagonal components in the low-energy or long-distance region, except for the renormalization of the remaining diagonal fields. This result strongly supports the infrared Abelian dominance conjectured by Ezawa and Iwazaki [3].

The dynamical mass generation of the off-diagonal components is understood based on the argument of Coleman-Weinberg type. See the paper[20]. The off-diagonal gluon propagator is given by

$$\langle A_\mu^a(x) A_\nu^b(y) \rangle = \int \frac{d^4k}{(2\pi)^4} e^{ikx} D_{\mu\nu}^{ab}(k), \quad (2.38)$$

$$D_{\mu\nu}^{ab}(k) := \delta^{ab} D_{\mu\nu}(k), \quad D_{\mu\nu}(k) := \frac{1}{k^2 - M_A^2} \left[g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 - \alpha M_A^2} \right] \quad (2.39)$$

The mass M_A of the off-diagonal gluon comes from the ghost–anti-ghost condensation caused by the four-ghost interaction. Especially, in the SU(2) case, we obtain⁴

$$M_A^2 = g^2 \langle i \bar{C}^a C^a \rangle = \frac{g^2}{16\pi} 4\pi e^{1-\gamma_E} \mu^2 \exp \left[\frac{-16\pi^2}{b_0 g^2(\mu)} \right] = \frac{g^2}{16\pi} 4\pi e^{1-\gamma_E} \Lambda_{QCD}^2. \quad (2.40)$$

The SU(3) case is more complicated, see [20] for details.

2.6 Step 5: Low-energy effective theory for diagonal fields

We are going to calculate the VEV of the *Abelian* components in the given non-Abelian gauge theory. If the non-Abelian gauge theory can be rewritten into the effective *Abelian* gauge theory which is expressed exclusively in terms of the Abelian

⁴Here we have used the minimal subtraction scheme (MS) in the dimensional regularization.

components only, we can calculate the above VEV in the resulting effective Abelian theory.

In the previous work [11], the author has derived an effective Abelian gauge theory by integrating out the off-diagonal components. The resulting theory was called the Abelian-projected effective gauge theory (APEGT). The APEGT is expected to be able to describe the low-energy region of gluodynamics or QCD. In order to obtain APEGT, we have introduced an antisymmetric tensor field $B_{\mu\nu}$ which enables us to derive the dual (magnetic) theory which is expected to be more efficient for describing the low-energy region. The magnetic theory can be obtained by the electro-magnetic duality transformation from the electric theory and vice versa. We have imposed the following duality in the tree level,

$$\boxed{B_{\mu\nu}^i \leftrightarrow *(\rho f_{\mu\nu}^i + \sigma g f^{iab} A_\mu^a A_\nu^b) := *Q_{\mu\nu}} \quad (2.41)$$

where $*$ denotes the Hodge star operation (or duality transformation) [33, 34] defined by

$$*Q_{\mu\nu} := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} Q^{\rho\sigma}. \quad (2.42)$$

First, the Yang-Mills Lagrangian is decomposed as

$$\mathcal{L}_{\text{YM}} = \mathcal{L}_{\text{YM}}^{(i)} + \mathcal{L}_{\text{YM}}^{(a)}, \quad \mathcal{L}_{\text{YM}}^{(i)} = -\frac{1}{4} (\mathcal{F}_{\mu\nu}^i)^2, \quad \mathcal{L}_{\text{YM}}^{(a)} = -\frac{1}{4} (\mathcal{F}_{\mu\nu}^a)^2. \quad (2.43)$$

The first piece is expanded as

$$\begin{aligned} \mathcal{L}_{\text{YM}}^{(i)} &= -\frac{1}{4} [f_{\mu\nu}^i + g f^{ibc} A_\mu^b A_\nu^c]^2 \\ &= -\frac{1}{4} (f_{\mu\nu}^i)^2 - \frac{g}{2} f_{\mu\nu}^i f^{ibc} A^{\mu b} A^{\nu c} - \frac{g^2}{4} (f^{ibc} A_\mu^b A_\nu^c)^2. \end{aligned} \quad (2.44)$$

The simplest form satisfying the duality requirement (2.41) is given by

$$\begin{aligned} \mathcal{L}_{\text{YM}}^{(i)} &= -\frac{1-\rho^2}{4} (f_{\mu\nu}^i)^2 - \frac{1-\rho\sigma}{2} g f_{\mu\nu}^i f^{ibc} A^{\mu b} A^{\nu c} - \frac{1-\sigma^2}{4} g^2 (f^{ibc} A_\mu^b A_\nu^c)^2 \\ &\quad - \frac{1}{4} (B_{\mu\nu}^i)^2 + \frac{i}{2} B_{\mu\nu}^i * Q^{\mu\nu i}. \end{aligned} \quad (2.45)$$

On the other hand, by defining the covariant derivative with respect to the Abelian gauge field,

$$D_\mu \Phi^A := (\partial_\mu \delta^{AB} + g f^{AiB} A_\mu^i) \Phi^B, \quad (2.46)$$

the second piece is rewritten as

$$\mathcal{L}_{\text{YM}}^{(a)} = -\frac{1}{4} [D_\mu A_\nu^a - D_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c]^2. \quad (2.47)$$

The tensor field $B_{\mu\nu}$ is introduced in such a way that $B_{\mu\nu}$ -integration recovers the original Yang-Mills theory. $B_{\mu\nu}$ is an auxiliary field, since it doesn't have the

corresponding kinetic term. However, the duality requirement leads to ambiguities for the identification as to what is the dual of $B_{\mu\nu}$. In fact, existence of two parameters ρ, σ in (2.41) reflects this ambiguity.

In particular, when $\rho = \sigma$, $Q_{\mu\nu}^i$ is nothing but the diagonal component of the non-Abelian field strength, $Q_{\mu\nu}^i = \rho \mathcal{F}_{\mu\nu}^i$, and hence, $B_{\mu\nu}^i = i\rho * \mathcal{F}_{\mu\nu}^i$. In view of this, the choice (2.41) is a generalization of that of the previous paper [11] in which two special cases, $\rho = \sigma = 1$ and $\rho = 0, \sigma = 1$ for $SU(2)$ have been discussed as eq. (2.9) and eq. (2.12) respectively [11]. The latter case has been first discussed by Quandt and Reinhardt[17]. In the $SU(2)$ case where $f^{abc} = 0$, the choice $\sigma = 1$ completely eliminates the quartic self-interaction among off-diagonal gluons.

The derivation of the APEGT was improved recently [35] so as to obtain the APEGT in the systematic way to the desired order where we have required the renormalizability of the resulting effective Abelian gauge theory as a guiding principle. We will discuss how to choose ρ and σ in subsection 3.4.

The strategy of deriving the APEGT is not unique. A way for obtaining the APEGT is to separate each field Φ^A into the high-energy mode $\tilde{\Phi}^A$ and the low-energy mode $\bar{\Phi}^A$ (i.e., $\Phi^A = \tilde{\Phi}^A + \bar{\Phi}^A$) and then to integrate out the *high-energy modes*,

$$\tilde{a}_\mu^i, \tilde{A}_\mu^a, \tilde{B}_{\mu\nu}, \tilde{C}^i, \tilde{\bar{C}}^i, \tilde{C}^a, \tilde{\bar{C}}^a,$$

of all the fields according to the idea of the Wilsonian renormalization group (RG). The resulting theory will be written in terms of the *low-energy modes* $\bar{\Phi}^A$. However, we can neglect the low-energy modes of A_μ^a and C^a, \bar{C}^a due to Abelian dominance and the final theory can be written in terms of the low-energy modes, $a_\mu^i, B_{\mu\nu}, C^i, \bar{C}^i$. In other words, the off-diagonal components A_μ^a and C^a, \bar{C}^a have only the high-energy modes. To one-loop level, the high-energy modes $\tilde{a}_\mu^i, \tilde{B}_{\mu\nu}, \tilde{C}^i, \tilde{\bar{C}}^i$ of the diagonal components don't contribute to the results. Therefore, we can identify A_μ^a and C^a, \bar{C}^a with the high-energy modes to be integrated out for obtaining the LEET. This strategy was adopted in the paper [35]. We do not adopt this method in this paper.

Another way is to integrate out all the massive fields

$$A_\mu^a, C^a, \bar{C}^a. \tag{2.48}$$

Then the resultant theory will be written in terms of the massless or light fields $a_\mu^i, B_{\mu\nu}, C^i, \bar{C}^i$. The effect of the massive fields will appear only through the renormalization of the resultant theory. This is an example of the decoupling theorem [18]. The only role of the heavy fields in the low momentum behavior of graphs without external heavy fields is their contribution to coupling constant and field-strength renormalization. The heavy fields effectively decouple and the low-momentum behavior of the theory is described by a renormalizable Lagrangian consisting of the massless fields only. The decoupling theorem applies not only to theories with massless fields but in fact to any renormalizable theory with different mass scales. At momentum smaller compared to the larger masses, the dynamics is determined by the light sector of the theory. In this paper we adopt this strategy. See also section 4.2.

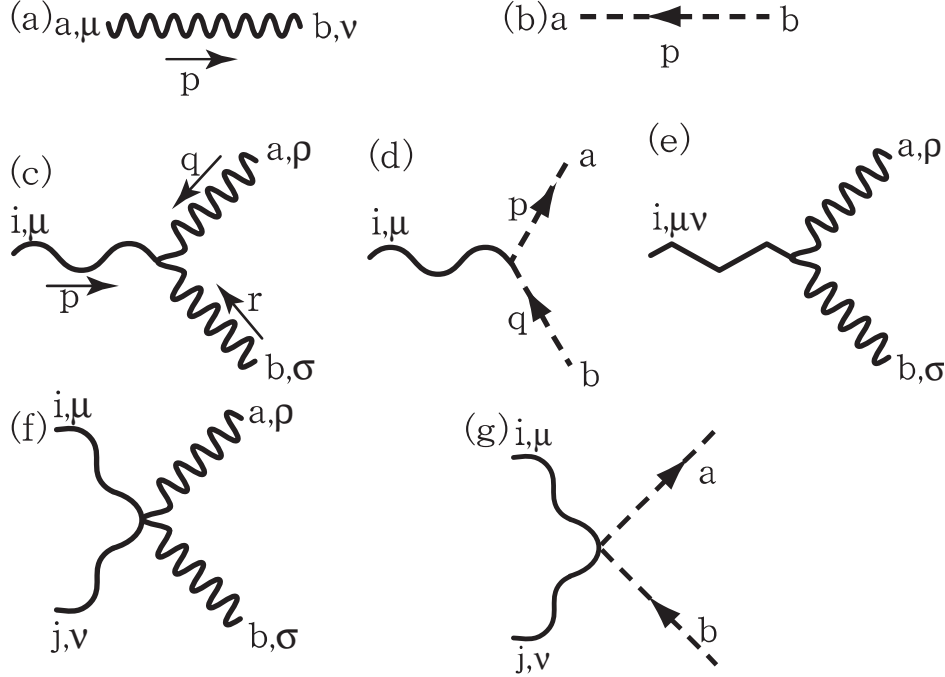


Figure 1: Feynman rules. (a),(b): off-diagonal propagators. The (rapidly vibrating) wavy line denotes the off-diagonal gluon A_μ^a and the broken line the ghost C^a or anti-ghost \bar{C}^a . (c),(d),(e): three-point vertices, (f),(g): four-point vertices. The (slowly vibrating) wavy line corresponds to the diagonal gluon a_μ^i , while the zig-zag line to the (diagonal) anti-symmetric tensor field $*B_{\mu\nu}^i$.

Consequently, the APEG T which was heuristically obtained in the paper [11] and improved systematically in the paper [35] is further modified by taking into account the mass of off-diagonal field components. The simplest derivation of the modified APEG T is to replace the massless off-diagonal propagator given in the previous paper [35] with the massive off-diagonal propagator in the QCD vacuum with ghost condensation.⁵

The Feynman rules are given as follows, see Fig. 1. We enumerate only a part of the rules which are necessary for the renormalization at one-loop level. The two-loop result will be reported in a subsequent paper[36].

2.6.1 Feynman rules

Propagators:

⁵However, the following steps can be performed irrespective of the origin of the off-diagonal gluon mass.

(a) Off-diagonal gluon propagators:

$$iD_{\mu\nu}^{ab} = -\frac{i}{p^2 - M_A^2} \left[g_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2 - \alpha M_A^2} \right] \delta^{ab}. \quad (2.49)$$

(b) Off-diagonal ghost propagators:⁶

$$i\Delta^{ab} = -\frac{(k^2 + v_d)\delta^{ab} + v_o\epsilon^{ab}}{(-k^2 - v_d)^2 + v_o^2}. \quad (2.50)$$

Three-point vertices:

(c) one diagonal and two off-diagonal gluons:

$$\begin{aligned} & i \left\langle a_\mu^i(p) A_\rho^a(q) A_\sigma^b(r) \right\rangle_{\text{bare}} \\ &= g f^{iab} \left[\left[(q - r)_\mu + \left\{ r - (1 - \rho\sigma)p + \frac{q}{\alpha} \right\}_\rho + \left\{ (1 - \rho\sigma)p - q - \frac{r}{\alpha} \right\}_\sigma \right] \right] \end{aligned} \quad (2.51)$$

where we have introduced the abbreviated notation,

$$\llbracket A_\mu + B_\rho + C_\sigma \rrbracket = A_\mu g_{\rho\sigma} + B_\rho g_{\sigma\mu} + C_\sigma g_{\mu\rho}. \quad (2.52)$$

(d) diagonal gluon, off-diagonal ghost and anti-ghost:

$$i \left\langle a_\mu^i \bar{C}^a(p) C^b(q) \right\rangle_{\text{bare}} = i(p + q)_\mu g f^{aib}. \quad (2.53)$$

(e) diagonal tensor and two off-diagonal gluons:

$$i \left\langle *B_{\mu\nu}^i A_\rho^a A_\sigma^b \right\rangle_{\text{bare}} = i2\sigma g I_{\mu\nu,\rho\sigma} f^{iab}, \quad (2.54)$$

where

$$I_{\mu\nu,\alpha\beta} := \frac{1}{2}(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}). \quad (2.55)$$

Four-point vertices:

(f) two diagonal gluons and two off-diagonal gluons:

$$i \left\langle a_\mu^i a_\nu^j A_\rho^a A_\sigma^b \right\rangle_{\text{bare}} = ig^2 f^{aic} f^{cjb} \left[2g_{\mu\nu}g_{\rho\sigma} - \left(1 - \frac{1}{\alpha}\right) (g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}) \right]. \quad (2.56)$$

(g) two diagonal gluons, off-diagonal ghost and anti-ghost:

$$i \left\langle a_\mu^i a_\nu^j \bar{C}^a C^b \right\rangle_{\text{bare}} = -2g^2 f^{aic} f^{cjb} g_{\mu\nu}. \quad (2.57)$$

⁶ This is the ghost propagator for $G = SU(2)$. For $G = SU(3)$, see the paper [20]. In this paper, however, we don't need the explicit form of the ghost propagator in the condensed vacuum. The details will be given in a forthcoming paper [37].

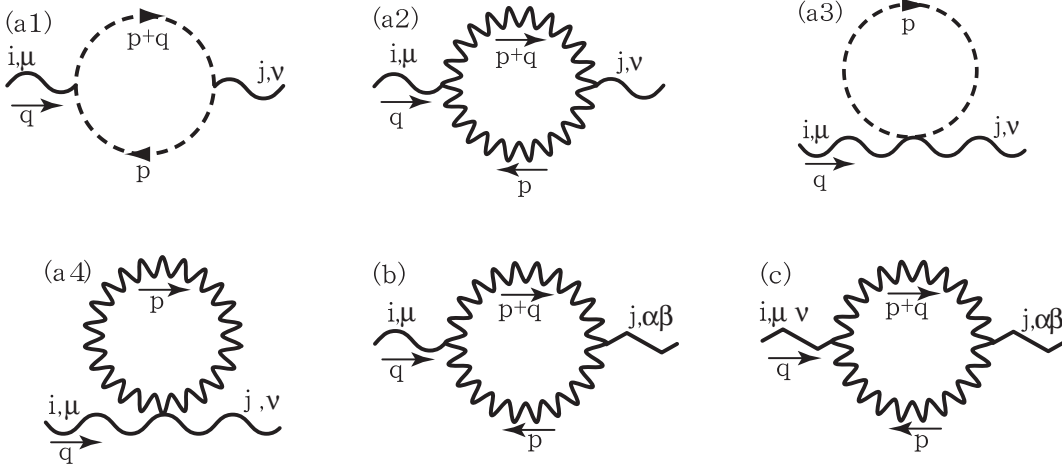


Figure 2: Vacuum polarization graphs which are necessary to obtain the APEGT.

2.6.2 APEGT

Thus the modified APEGT is written as⁷

$$\mathcal{L}_{APEGT} = \mathcal{L}_{APEGT}^0 + \delta_{(1)} \mathcal{L}_{APEGT}, \quad (2.58)$$

where \mathcal{L}_{APEGT}^0 is the bare diagonal part,

$$\mathcal{L}_{APEGT}^0 = -\frac{1-\rho^2}{4} (f_{\mu\nu}^i)^2 - \frac{1}{2} \rho * B_{\mu\nu}^i f^{\mu\nu i} - \frac{1}{4} (B_{\mu\nu}^i)^2, \quad (2.59)$$

and the quantum part $\delta_{(1)} \mathcal{L}_{APEGT}$ is obtained by calculating the vacuum polarization graphs in Fig. 2. In consistent with the renormalizability of the diagonal part,

$$\delta_{(1)} \mathcal{L}_{APEGT} = -\frac{\delta_1}{4} (f_{\mu\nu}^i)^2 - \frac{\delta_2}{2} * B_{\mu\nu}^i f^{\mu\nu i} - \frac{\delta_3}{4} (B_{\mu\nu}^i)^2 + O\left(\frac{p^2}{M_A^2}\right), \quad (2.60)$$

where $O\left(\frac{p^2}{M_A^2}\right)$ is the modification term which is obtained in the next section. We use the dimensional regularization to determine δ_1, δ_2 and δ_3 which corresponds to Fig. 2(a), (b), (c) respectively. Apart from the finite part, the divergent part of δ_1 ,

⁷ In Fig. 2, we have omitted Feynman graphs which include external diagonal-ghost and diagonal-anti-ghost lines, and internal off-diagonal-ghosts (-anti-ghost) or internal off-diagonal-gluon lines. The diagonal ghost and diagonal anti-ghost comes from the GF+FP term (2.37) for the diagonal gluon a_μ^i . The higher-order terms with more than two external lines are suppressed by N^{-2} in the large N expansion, as shown in section 4.1. If we neglect such higher-order terms, the bilinear terms in the external ghost and anti-ghost decouple from the (2.58). Therefore, we don't discuss the contribution from diagonal ghosts and anti-ghosts to (2.58) in this paper.

δ_2 and δ_3 (proportional to ϵ^{-1} where $\epsilon := 2 - D/2$) is given by

$$\begin{aligned}
\delta_1 &= \delta_{a1} + \delta_{a2}, \\
\begin{cases} \delta_{a1} = \left[(2 - \rho_R \sigma_R)^2 - \frac{2}{3} + \frac{1 - \alpha_R}{2} (2 - \rho_R \sigma_R) \rho_R \sigma_R \right] \frac{(\mu^{-\epsilon} g_R)^2 C_2(G)}{(4\pi)^2 \epsilon}, \\ \delta_{a2} = \frac{1}{3} \frac{(\mu^{-\epsilon} g_R)^2 C_2(G)}{(4\pi)^2 \epsilon}, \end{cases} \\
\delta_2 &= \left[\sigma_R (2 - \rho_R \sigma_R) - \frac{1 - \alpha_R}{2} \sigma_R (1 - \rho_R \sigma_R) \right] \frac{(\mu^{-\epsilon} g_R)^2 C_2(G)}{(4\pi)^2 \epsilon}, \\
\delta_3 &= -\frac{1 + \alpha_R}{2} \sigma_R^2 \frac{(\mu^{-\epsilon} g_R)^2 C_2(G)}{(4\pi)^2 \epsilon}, \tag{2.61}
\end{aligned}$$

where δ_{a1} and δ_{a2} are the contributions from the graphs (a1) and (a2) in Fig. 2 respectively, and $C_2(G)$ is a quadratic Casimir operator defined by $C_2(G)\delta^{AB} = f^{ACD}f^{BCD}$ ($C_2(G) = N$ for $G = SU(N)$).⁸

Note that the Lagrangian \mathcal{L}_{APEGT}^0 is bilinear in the diagonal fields $f_{\mu\nu}$ and $B_{\mu\nu}$. So is the term $\delta_{(1)}\mathcal{L}_{APEGT}$. Therefore, the divergence can be absorbed if we renormalize the theory. The mass generation of off-diagonal components also justifies neglecting the higher-derivative terms in the APEGT in the low-energy region. The additional term of order $O(p^2/M_A^2)$ is discussed in the next step. In the $SU(N)$ case, it is shown that the higher-order terms which is quartic and more in the fields can be neglected within the framework of the large N expansion, see section 4.2.

A non-trivial consequence is that the β -function in the APEGT has exactly the same form as that in the original Yang-Mills theory and is independent of the gauge parameter α and two parameters ρ, σ ,

$$\beta(g) := \mu \frac{dg(\mu)}{d\mu} = -b_0 g^3(\mu) + O(g^5), \quad b_0 = \frac{11}{3}N > 0. \tag{2.62}$$

2.7 Step 6: Dynamical generation of the kinetic term of $B_{\mu\nu}$

The antisymmetric tensor field $B_{\mu\nu}$ was introduced as an auxiliary field, since it has no kinetic term. In this subsection, we show that the $B_{\mu\nu}$ can have its kinetic term as a consequence of radiative corrections, i.e., *the kinetic term of $B_{\mu\nu}$ is dynamically generated*. It turns out that *the dynamical generation of the kinetic terms occurs only when the off-diagonal gluons are massive*. This fact plays the most important role in deriving the dual superconductivity.

We proceed to calculate the vacuum polarization for the $B_{\mu\nu}$ field up to one-loop of the massive off-diagonal gluons, see Fig.2(c). Following the Feynmann rule given in Fig. 1, the vacuum polarization of Fig. 2(c) is written in momentum space as

$$\begin{aligned}
\Pi_{\mu\nu, \alpha\beta}^{ij}(k) &:= \frac{1}{2} \int \frac{d^4 p}{(2\pi)^4} D_{\sigma_1 \sigma_2}(p) \delta^{d_1 d_2} [-2\sigma g f^{ic_1 d_1} I_{\mu\nu, \rho_1 \sigma_1}] \\
&\quad \times D_{\rho_1 \rho_2}(p+k) \delta^{c_1 c_2} [-2\sigma g f^{jc_2 d_2} I_{\alpha\beta, \rho_2 \sigma_2}], \tag{2.63}
\end{aligned}$$

⁸ For a special choice of the parameters, $\rho = \sigma = 0$ and $\alpha = 0$, only the δ_1 has been calculated by Quandt and Reinhardt[17] for $SU(2)$.

where

$$D_{\mu\nu}(k) := \frac{1}{k^2 - M_A^2} \left[g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 - \alpha M_A^2} \right] \quad (2.64)$$

$$= \frac{1}{k^2 - M^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right) + \frac{k_\mu k_\nu}{M^2} \frac{1}{k^2 - \alpha M^2}. \quad (2.65)$$

Hence the additional contribution coming from Fig. 2(c) to the APEGT is given by

$$\delta_{(1)}^e \mathcal{L}_{APEGT} = \int \frac{d^4 k}{(2\pi)^4} * B_{\mu\nu}^i(k) \Pi_{\mu\nu,\alpha\beta}^{ij}(k) * B_{\alpha\beta}^j(-k). \quad (2.66)$$

By making use of the MS scheme in the dimensional regularization method, we have arrived the following expression after straightforward but tedious calculations:
 $\epsilon := 2 - D/2$

$$\begin{aligned} \Pi_{\mu\nu,\alpha\beta}^{ij}(k) &= \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} I_{\mu\nu,\alpha\beta} \left[\epsilon^{-1} \frac{1+\alpha}{2} \right] \\ &+ \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} I_{\mu\nu,\alpha\beta} \int_0^1 dx \left\{ \left[-x(1-x) \frac{k^2}{M_A^2} \right] \ln \left[\frac{M_A^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \right. \\ &- \left[\alpha + (1-\alpha)x - x(1-x) \frac{k^2}{M_A^2} \right] \ln \left[\{ \alpha + (1-\alpha)x \} \frac{M_A^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \\ &+ (\gamma_E - 1)(1-\alpha)(1-x) - \gamma_E + \ln 4\pi \left. \right\} \\ &+ \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} \frac{1}{2} \frac{k^2}{M_A^2} (I - P)_{\mu\nu,\alpha\beta} [\dots] + O(\epsilon), \end{aligned} \quad (2.67)$$

where $C_2 \delta^{ij} \equiv f^{iAB} f^{jAB} = f^{iab} f^{jab}$ and γ_E is Euler's constant $\gamma_E = 0.5772 \dots$. See Appendix C for complete expression.

In the neighborhood of $k^2 = 0$, i.e, in the low-energy region such that $k^2/M^2 \ll 1$, we use the *low-energy* (or *large mass*) expansion, e.g.,

$$\begin{aligned} \ln \left[\frac{M_A^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] &= \ln \frac{M_A^2}{\mu^2} + \ln \left[1 - x(1-x) \frac{k^2}{M_A^2} \right] \\ &= \ln \frac{M_A^2}{\mu^2} - \sum_{n=1}^{\infty} \frac{1}{n} x^n (1-x)^n \left(\frac{k^2}{M_A^2} \right)^n. \end{aligned} \quad (2.68)$$

Note that this expansion is possible only when $M_A \neq 0$. This is a reason why the dynamical generation of the kinetic term takes place. Thus we obtain

$$\begin{aligned} &\Pi_{\mu\nu,\alpha\beta}^{ij}(k) \\ &= -\frac{2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} \left\{ I_{\mu\nu,\alpha\beta} \left[\epsilon^{-1} \frac{1+\alpha}{2} + f_0(\alpha) + f_1(\alpha) \frac{k^2}{M_A^2} + f_2(\alpha) \frac{k^4}{M_A^4} \right] \right. \\ &\quad \left. + \frac{1}{2} \frac{k^2}{M_A^2} (I - P)_{\mu\nu,\alpha\beta} \left[h_0(\alpha) + h_1(\alpha) \frac{k^2}{M_A^2} \right] \right\} + O \left(\frac{k^6}{M_A^6} \right), \end{aligned} \quad (2.69)$$

where

$$f_0(\alpha) := -\frac{1+\alpha}{2} \ln \frac{M_A^2}{\mu^2} + (\gamma_E - \ln 4\pi - 1)(1-\alpha)\frac{1}{2} - \gamma_E + \ln 4\pi \\ + \frac{1}{1-\alpha} \left[\frac{\alpha^2}{2} \ln \alpha + \frac{1}{4} - \frac{1}{4}\alpha^2 \right], \quad (2.70)$$

$$f_1(\alpha) := \int_0^1 dx \, x(1-x) \{1 + \ln[\alpha + (1-\alpha)x]\} \quad (2.71)$$

$$= \frac{1}{6} + \frac{1}{(1-\alpha)^3} \left\{ \frac{\alpha^3}{3} \ln \alpha - \frac{\alpha^3}{9} + \frac{1}{9} - (1+\alpha) \left[\frac{\alpha^2}{2} \ln \alpha - \frac{\alpha^2}{4} + \frac{1}{4} \right] \right. \\ \left. + \alpha(\alpha \ln \alpha - \alpha + 1) \right\}, \quad (2.72)$$

$$f_2(\alpha) := \int_0^1 dx \, \left[x^2(1-x)^2 - \frac{1}{2} \frac{x^2(1-x)^2}{\alpha + (1-\alpha)x} \right] \quad (2.73)$$

$$= \frac{1}{30} - \frac{1}{(1-\alpha)^5} \left[\frac{1}{24} - \frac{1}{3}\alpha + \frac{1}{3}\alpha^3 - \frac{1}{24}\alpha^4 - \frac{1}{2}\alpha^2 \ln \alpha \right]. \quad (2.74)$$

The function $f = f_1(\alpha)$ is a monotonically increasing function in α defined for $\alpha > 0$ and positive $f_1(\alpha) > 0$ when $\alpha > 0$. In particular, $f_1(0) = 0.0278$, $f_1(1) = 1/6$, and $f_1(11/3) = 0.302$. On the other hand, $f_2(\alpha)$ is a monotonically increasing function defined for $\alpha > 0$. Hence, $0 < f_2(\alpha) < 1/30$ for $\alpha_0 < \alpha < +\infty$ and $-0.00833 < f_2(\alpha) < 0$ for $0 < \alpha < \alpha_0$ with $\alpha_0 \cong 0.12$. Incidentally, $f_2(1) = 1/60$, $f_2(2) = 0.0220$, $f_2(11/3) = 0.0258$.

In the following derivation, we don't need the explicit form of the functions $h_i(\alpha)$. This is because it can be shown that the term proportional to $(I - P)$ (the so-called boundary term) does not contribute to the area law. Moreover, if we impose the gauge fixing condition $\partial^\mu * B_{\mu\nu} = 0$, then such terms give vanishing contributions. See Appendix C and a subsequent paper [72] for more details.

In the configuration space, therefore, we obtain ⁹

$$\delta_{(1)}^c \mathcal{L}_{APEGT} = \int d^4x \frac{C_2 \sigma^2 g^2}{8\pi^2} * B_{\mu\nu}^i(x) I^{\mu\nu, \alpha\beta} \left[f_0(\alpha) - f_1(\alpha) \frac{\partial_\lambda \partial^\lambda}{M_A^2} + f_2(\alpha) \frac{(\partial_\lambda \partial^\lambda)^2}{M_A^4} \right] * B_{\alpha\beta}^i(x). \quad (2.75)$$

The first term in the RHS of $\delta_{(1)}^c \mathcal{L}_{APEGT}$ gives a piece of the counter terms of the \mathcal{L}_{APEGT}^0 (together with a finite part of δ_3) and the remaining parts give new terms which can not be absorbed into the bare part, the non-renormalizable contribution. In the limit $M_A \rightarrow \infty$, the non-renormalizable terms disappear. It is very important to notice that *the kinetic term for the auxiliary tensor field $B_{\mu\nu}$ is generated due to radiative corrections* and hence $B_{\mu\nu}$ is regarded as *the massive propagating field (at least in the low-energy region less than M_A)*. This is one of the main results of this paper. The implications of this fact to low-energy QCD will be discussed in what follows. The other contributions from Fig. 2(a),(b) can also be evaluated in the similar

⁹For an antisymmetric tensor $A_{\mu\nu}$, it is easy to see $I_{\mu\nu, \alpha\beta} A^{\alpha\beta} = A_{\mu\nu}$ and $A^{\mu\nu} I_{\mu\nu, \alpha\beta} = A_{\alpha\beta}$.

manner. Explicit form will be given in a subsequent paper, since we don't need them in this paper.

Thus, we have obtained the improved version of the APEGT where the VEV of the functional $f[a, h]$ of a, h is given by

$$\langle f[a, h] \rangle_{APEGT} = Z_{APEGT}^{-1} \int \mathcal{D}a_\mu^i \mathcal{D}h_{\mu\nu}^i \exp \{i S_{APEGT}[a, h]\} f[a, h], \quad (2.76)$$

using the action

$$S_{APEGT}[a, h] = \int d^4x \left[\mathcal{L}_e[a] + \frac{1}{2} \rho K^{-1/2} h_{\mu\nu}^i f^{\mu\nu i} + \mathcal{L}_d[h] \right], \quad (2.77)$$

$$\mathcal{L}_e[a] := -\frac{1-\rho^2}{4} (f_{\mu\nu}^i)^2 + O\left(\frac{f^4}{M_A^4}\right), \quad (2.78)$$

$$\mathcal{L}_d[h] := -\frac{1}{4} (h_{\mu\nu}^i)^2 - \frac{1}{4\eta^2} h_{\mu\nu}^i \partial_\lambda \partial^\lambda h^{\mu\nu i} + \frac{1}{4\gamma^4} h_{\mu\nu}^i (\partial_\lambda \partial^\lambda)^2 h^{\mu\nu i} + O\left(\frac{(\partial^2)^3}{M_A^6}\right), \quad (2.79)$$

where we have introduced the rescaling factor K and the rescaled tensor field $h_{\mu\nu}$ by

$$K := 1 + \frac{C_2 \sigma^2 g^2}{2\pi^2} f_0(\alpha) > 0, \quad (2.80)$$

$$h_{\mu\nu}^i(x) := K^{1/2} * B_{\mu\nu}^i(x), \quad (2.81)$$

and two quantities η and γ with mass dimension by

$$\eta^2 := f_1(\alpha)^{-1} \frac{2\pi^2}{g^2 C_2 \sigma^2} K M_A^2, \quad (2.82)$$

$$\gamma^4 := f_2(\alpha)^{-1} \frac{2\pi^2}{g^2 C_2 \sigma^2} K M_A^4. \quad (2.83)$$

The action $S_{APEGT}[a, h]$ has U(1) invariance of $a_\mu \rightarrow a_\mu + \partial_\mu \theta$, i.e.,

$$S_{APEGT}[a + d\theta, h] = S_{APEGT}[a, h]. \quad (2.84)$$

Note that the Lagrangian of the modified APEGT is still bilinear in the fields, $f_{\mu\nu}$ and $B_{\mu\nu}$.

2.8 Step 7: Dual transformations

We use an idea inspired by the (Abelian) electric-magnetic duality to calculate the Abelian field correlators in the cumulant expansion (2.25). Note that the Abelian diagonal field correlation functions $\langle f_{\mu\nu}^\xi(x) f_{\rho\sigma}^\xi(y) \rangle_{YM}$ can be calculated in the APEGT, since the APEGT is obtained by integrating out all off-diagonal components. Hence we have

$$\langle f_{\mu\nu}^\xi(x) f_{\rho\sigma}^\xi(y) \rangle_{YM} = \langle f_{\mu\nu}^\xi(x) f_{\rho\sigma}^\xi(y) \rangle_{APEGT}. \quad (2.85)$$

Then the expectation value of the Wilson loop obeys

$$\begin{aligned} & \langle W(C) \rangle_{YM} \\ &= \int d\mu_C(\xi) \exp \left[-\frac{J^2 g^2}{2} \int_{S_C} dS^{\mu\nu}(x) \int_{S_C} dS^{\rho\sigma}(y) \langle f_{\mu\nu}^\xi(x) f_{\rho\sigma}^\xi(y) \rangle_{APEGT} + \dots \right] \end{aligned} \quad (2.86)$$

We use repeatedly the integration by parts to rewrite the expectation value of the electric quantity into that of the magnetic quantity as follows:¹⁰

$$\begin{aligned} & Z_{APEGT} \langle f^{\mu\nu}(x) f^{\rho\sigma}(y) \rangle_{APEGT} \\ &= \int \mathcal{D}a_\mu \mathcal{D}h_{\mu\nu} \exp \left\{ i \int d^4x (\mathcal{L}_e[a] + \mathcal{L}_d[h]) \right\} \\ & \quad \times \left(\frac{2K^{1/2}}{i\rho} \right)^2 \frac{\delta}{\delta h_{\mu\nu}(x)} \frac{\delta}{\delta h_{\rho\sigma}(y)} \exp \left\{ i \int d^4x \frac{1}{2} \rho K^{-1/2} h_{\mu\nu} f^{\mu\nu} \right\} \\ &= \int \mathcal{D}a_\mu \mathcal{D}h_{\mu\nu} \exp \left\{ i \int d^4x \frac{1}{2} \rho K^{-1/2} h_{\mu\nu} f^{\mu\nu} \right\} \left(\frac{2K^{1/2}}{i\rho} \right)^2 \\ & \quad \times \frac{\delta}{\delta h_{\rho\sigma}(y)} \frac{\delta}{\delta h_{\mu\nu}(x)} \exp \left\{ i \int d^4x (\mathcal{L}_e[a] + \mathcal{L}_d[h]) \right\} \\ &= \int \mathcal{D}a_\mu \mathcal{D}h_{\mu\nu} \exp \left\{ i \int d^4x \left(\frac{1}{2} \rho K^{-1/2} h_{\mu\nu} f^{\mu\nu} + \mathcal{L}_e[a] \right) \right\} \left(\frac{2K^{1/2}}{i\rho} \right)^2 \\ & \quad \times \frac{\delta}{\delta h_{\rho\sigma}(y)} \frac{\delta}{\delta h_{\mu\nu}(x)} \exp \left\{ i \int d^4x \mathcal{L}_d[h] \right\}. \end{aligned} \quad (2.87)$$

The functional derivatives are performed as

$$\begin{aligned} & \frac{\delta}{\delta h_{\rho\sigma}(y)} \frac{\delta}{\delta h_{\mu\nu}(x)} \exp \left\{ i \int d^4x \mathcal{L}_d[h] \right\} \\ &= \frac{\delta}{\delta h_{\rho\sigma}(y)} \left(\frac{1}{i} \mathcal{D}[\partial_x] h_{\mu\nu}(x) \exp \left\{ i \int d^4x \mathcal{L}_d[h] \right\} \right) \\ &= \left[\frac{1}{i} \mathcal{D}[\partial_x] \delta(x-y) (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\nu\rho} \delta_{\mu\sigma}) + \left(\frac{1}{i} \right)^2 \mathcal{D}[\partial_x] h_{\mu\nu}(x) \mathcal{D}[\partial_y] h_{\rho\sigma}(y) \right] \\ & \quad \times \exp \left\{ i \int d^4x \mathcal{L}_d[h] \right\}, \end{aligned} \quad (2.88)$$

¹⁰ Here we have used a property of the measure $\mathcal{D}h$, $\int \mathcal{D}h \frac{\delta}{\delta h}(\dots) = 0$.

where we have defined an operator,

$$\mathcal{D}[\partial] := 1 + \frac{\partial^2}{\eta^2} - \frac{\partial^4}{\gamma^4}. \quad (2.89)$$

Here $\mathcal{D}[\partial]$ denotes quantum corrections for the h field.

Therefore, we obtain the relationship reflecting the duality between electric and magnetic sector,

$$\begin{aligned} & \langle f_{\mu\nu}(x) f_{\rho\sigma}(y) \rangle_{APEG T} \\ &= \left(\frac{2K^{1/2}}{i\rho} \right)^2 \left(\frac{1}{i} \right) \mathcal{D}[\partial_x] \delta(x-y) (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\nu\rho} \delta_{\mu\sigma}) \\ &+ \left(\frac{2K^{1/2}}{i\rho} \right)^2 \left(\frac{1}{i} \right)^2 \langle \mathcal{D}[\partial_x] h_{\mu\nu}(x) \mathcal{D}[\partial_x] h_{\rho\sigma}(y) \rangle_{APEG T}. \end{aligned} \quad (2.90)$$

Thus the VEV of the Wilson loop in the Yang-Mills theory is rewritten in terms of the correlation functions of the magnetic quantity in the APEG T as

$$\begin{aligned} \langle W(C) \rangle_{YM} &= \int d\mu_C(\xi) \exp \left[-2J^2 g^2 \rho^{-2} K \int_{S_C} dS^{\mu\nu}(x) \int_{S_C} dS^{\rho\sigma}(y) \right. \\ &\quad \left. \times \langle \mathcal{D}[\partial_x] h_{\mu\nu}^\xi(x) \mathcal{D}[\partial_y] h_{\rho\sigma}^\xi(y) \rangle_{APEG T} + \dots \right]. \end{aligned} \quad (2.91)$$

This is identified as the cumulant expansion of

$$\langle W(C) \rangle_{YM} = e^{[\dots]} \int d\mu_C(\xi) \left\langle \exp \left[2i\rho^{-1} K^{1/2} Jg \int_{S_C} dS^{\mu\nu}(x) \mathcal{D}[\partial_x] h_{\mu\nu}^\xi(x) \right] \right\rangle_{APEG T}, \quad (2.92)$$

where $[\dots]$ denotes the field-independent-constant part (a phase factor) in the first term in the RHS of (2.90) in which we have no interest. It should be remarked that the above result is totally independent from the explicit form of $\mathcal{L}_e[a]$, even if we include higher order corrections of low-energy expansions.

The above result implies that we can calculate the Wilson loop by making use of the magnetic theory which is written in terms of tensor field $h_{\mu\nu}$ alone. By performing the integration over a_μ field, we can obtain such a dual magnetic theory.

By including the gauge fixing term of a_μ^i into the action $S_{APEG T}[a, h]$,

$$\mathcal{L}_{GF}[a] := -\frac{1}{2\beta} (\partial^\mu a_\mu^i)^2, \quad (2.93)$$

the action $S_{APEG T}[a, h]$ reads

$$S_{APEG T}[a, h] = \int d^4x \left\{ \mathcal{L}_e^{tot}[a] - \rho K^{-1/2} a^{\nu i} \partial^\mu h_{\mu\nu}^i + \mathcal{L}_d[h] \right\}, \quad (2.94)$$

where¹¹

$$\begin{aligned} \mathcal{L}_e^{tot}[a] &:= \mathcal{L}_e[a] + \mathcal{L}_{GF}[a] \\ &= \frac{1}{2} a_\mu^i \left[(1 - \rho^2) g^{\mu\nu} \partial^2 - (1 - \rho^2 - \beta^{-1}) \partial^\mu \partial^\nu \right] a_\nu^i + O\left(\frac{f^4}{M_A^4}\right). \end{aligned} \quad (2.95)$$

¹¹ It should be understood that $\mathcal{L}_e^{tot}[a]$ is obtained after integrating out B^i, C^i, \bar{C}^i .

For $\rho \neq 1$, integrating out the a_μ field yields

$$\begin{aligned}\mathcal{L}_d[h]' &= \mathcal{L}_d[h] + \frac{\rho^2 K^{-1}}{2(1-\rho^2)} \partial^\tau h_{\tau\mu}^i \frac{1}{\partial^4} \left\{ g^{\mu\nu} \partial^2 - [1 - (1-\rho^2)\beta] \partial^\mu \partial^\nu \right\} \partial^\lambda h_{\lambda\nu}^i \\ &= \mathcal{L}_d[h] + \frac{\rho^2 K^{-1}}{2(1-\rho^2)} \partial^\tau h_{\tau\mu}^i \frac{1}{\partial^2} g^{\mu\nu} \partial^\lambda h_{\lambda\nu}^i.\end{aligned}\tag{2.96}$$

Thus the VEV of the Wilson operator is calculated from

$$\langle W(C) \rangle_{YM} = \int d\mu_C(\xi) \left\langle \exp \left[2i\rho^{-1} K^{1/2} Jg \int_{S_C} dS^{\mu\nu}(x) \mathcal{D}[\partial_x] h_{\mu\nu}^\xi(x) \right] \right\rangle_d \tag{2.97}$$

$$= Z_d^{-1} \int \mathcal{D}h_{\mu\nu}^i \exp \left\{ i \int d^4x \mathcal{L}_d[h]' \right\} \mathcal{J}[h], \tag{2.98}$$

where

$$\mathcal{J}[h] := \exp \left[2i\rho^{-1} K^{1/2} Jg \int_{S_C} dS^{\mu\nu}(x) \mathcal{D}[\partial_x] h_{\mu\nu}^\xi(x) \right]. \tag{2.99}$$

When $\rho = 1$, $\mathcal{L}_e[a]$ contains only the higher-order terms. In the limit $\rho \rightarrow 1$, the second term in (2.96) yields the constraint $\delta(\partial^\lambda h_{\lambda\nu}^i)$ in the measure,

$$\langle W(C) \rangle_{YM} = Z_d^{-1} \int \mathcal{D}h_{\mu\nu}^i \delta(\partial^\lambda h_{\lambda\nu}^i) \exp \left\{ i \int d^4x \mathcal{L}_d[h] \right\} \mathcal{J}[h]. \tag{2.100}$$

If we integrate out tensor field $h_{\mu\nu}$ in $S_{APEGT}[a, h]$, we will obtain the electric theory which is written in terms of the diagonal gluon field a_μ alone. This theory can lead to the area law too, as will be discussed in a subsequent paper [72].

2.9 Step 8: Recovery of hypergauge symmetry and gauge fixing

The action $S_{APEGT}[a, h]$ has the U(1) gauge invariance for the gauge transformation of the diagonal gluon field: $a_\mu \rightarrow a_\mu + \partial_\mu \theta$, i.e., $S_{APEGT}[a + d\theta, h] = S_{APEGT}[a, h]$. In this section we consider the symmetry for the tensor field h .

Now we introduce the field strength H of an antisymmetric tensor field h (the so-called Kalb-Ramond field), $H := dh$, i.e.,

$$H_{\mu\nu\lambda} := \partial_\lambda h_{\mu\nu} + \partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} \tag{2.101}$$

Note that the field strength H is invariant, $H_{\mu\nu\lambda} \rightarrow H_{\mu\nu\lambda}$, under the hypergauge transformation, $h \rightarrow h + d\zeta$, i.e.,

$$h_{\mu\nu} \rightarrow h_{\mu\nu}^\zeta := h_{\mu\nu} + \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu. \tag{2.102}$$

We require the invariance of the measure $\mathcal{D}h_{\mu\nu}$ under the hypergauge transformation. However, the Lagrangian $\mathcal{L}_{APEGT}[a, h]$ or $\mathcal{L}_d[h]'$ does not have the invariance under

the hypergauge transformation due to the existence of the mass term $(h_{\mu\nu})^2$. Nevertheless, we can recover the invariance by introducing new degrees of freedom¹² Λ_μ where Λ_μ transforms as

$$\Lambda_\mu \rightarrow \Lambda_\mu^\zeta := \Lambda_\mu - \zeta_\mu. \quad (2.103)$$

In fact, the combination $h^\Lambda := h + d\Lambda$, i.e.,

$$h_{\mu\nu}^\Lambda = h_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu, \quad (2.104)$$

is invariant under the combined transformations, (2.102) and (2.103). Therefore, the Lagrangian,

$$\mathcal{L}_m[h, \Lambda] := \mathcal{L}_m[h^\Lambda] = -\frac{1}{4}(h_{\mu\nu}^i + \partial_\mu \Lambda_\nu^i - \partial_\nu \Lambda_\mu^i)^2 + \frac{1}{12\eta^2}(H_{\mu\nu\lambda}^i)^2 + \frac{1}{4\gamma^4}(\partial^\lambda H_{\lambda\mu\nu}^i)^2, \quad (2.105)$$

is also invariant, i.e., $\mathcal{L}_m[h, \Lambda] = \mathcal{L}_m[h^\zeta, \Lambda^\zeta]$.

As we have recovered the hypergauge invariance, we need to fix the hypergauge invariance in quantizing the dual magnetic theory $\mathcal{L}_m[h, \Lambda]$. From this viewpoint, we adopt the condition,¹³

$$\partial^\nu h_{\mu\nu}^i = 0, \quad (2.106)$$

as a gauge fixing condition for the antisymmetric tensor field.¹⁴ Under this condition, the derivative terms of $h_{\mu\nu}$ in $\mathcal{L}_m[h^\Lambda]$ recover the corresponding terms of $\mathcal{L}_d[h]$,

$$(H_{\mu\nu\lambda}^i)^2 = -3h_{\mu\nu}^i \partial^2 h_{\mu\nu}^i - 6\partial_\mu h_{\mu\nu}^i \partial_\lambda h^{\nu\lambda i} \rightarrow -3h_{\mu\nu}^i \partial_\lambda \partial^\lambda h^{\mu\nu i}, \quad (2.107)$$

$$(\partial^\lambda H_{\lambda\mu\nu}^i)^2 = (\partial^\lambda \partial_\lambda h_{\mu\nu}^i + \partial_\mu \partial^\lambda h_{\nu\lambda}^i + \partial_\nu \partial^\lambda h_{\lambda\mu}^i)^2 \rightarrow h_{\mu\nu}^i (\partial_\lambda \partial^\lambda)^2 h^{\mu\nu i}. \quad (2.108)$$

Here it turns out that all the terms proportional to $(1 - P)$ in $\delta_{(1)}^c \mathcal{L}_{APEGT}$ vanish under this condition. Therefore, $\mathcal{L}_m[h, \Lambda]$ under the condition reduces to

$$\mathcal{L}_m[h, \Lambda] = -\frac{1}{4}(h_{\mu\nu}^i + \partial_\mu \Lambda_\nu^i - \partial_\nu \Lambda_\mu^i)^2 - \frac{1}{4\eta^2} h_{\mu\nu}^i \partial_\lambda \partial^\lambda h^{\mu\nu i} + \frac{1}{4\gamma^4} h_{\mu\nu}^i (\partial_\lambda \partial^\lambda)^2 h^{\mu\nu i}. \quad (2.109)$$

It is shown that the theory with $\mathcal{L}_m[h, \Lambda]$ reduces to the original theory given by $\mathcal{L}_d[h]$ by integrating out Λ field after fixing the gauge freedom of Λ_μ , see Appendix D.

Thus we obtain an alternative dual description of low-energy Gluodynamics in terms of $h_{\mu\nu}$ and Λ_μ . Especially, for $G = SU(2)$, we obtain

$$\langle W(C) \rangle_{YM} = Z_M^{-1} \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \int \mathcal{D}\Lambda_\mu \exp \{ i S_M[h^\Lambda; C] \}, \quad (2.110)$$

¹²This field plays the similar role to the Stückelberg field in the massive vector theory which recovers the gauge invariance of the vector field.

¹³In section 3.3 we discuss the relationship between this condition and the setting up of the previous paper [11].

¹⁴In the manifestly covariant quantization of the gauge theory, we need to introduce the ghost as is well known. However, it is not enough for the antisymmetric tensor gauge theory, since we need to introduce the ghost for ghost in order to completely fix the gauge degrees of freedom. Such a theory is called a reducible theory. In this subsection we treat the theory in a naive manner. However, the result is unchanged if we take into account the reducibility of the theory. See Appendix D.

where

$$S_M[h^\Lambda; C] = \int d^4x \mathcal{L}_m[h^\Lambda] + 2\rho^{-1} K^{1/2} Jg \int_{S_C} dS^{\mu\nu}(x) \mathcal{D}[\partial_x] h_{\mu\nu}(x). \quad (2.111)$$

Apart from the third term in $\mathcal{L}_m[h]$, the above action (2.105) coincides with the action of confining string proposed by Polyakov [25] in the weak field limit, see section 6.

2.10 Step 9: Change of variables (path-integral duality transformation)

We proceed to rewrite the LEET (2.111) into another form which is useful to derive the dual Ginzburg-Landau theory.

First of all, we rewrite the surface integral in $S_{APEGT}[h^\Lambda; C]$ into the volume integral as follows. Let $\sigma = (\sigma^1, \sigma^2)$ be two-dimensional coordinate on the surface which is bounded by the Wilson loop C . Then

$$\begin{aligned} \int_{S_C} dS^{\mu\nu}(x(\sigma)) \tilde{h}_{\mu\nu}(x(\sigma)) &= \frac{1}{2} \int d^2\sigma \epsilon^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} \tilde{h}_{\mu\nu}(x(\sigma)) \\ &= \frac{1}{2} \int d^2\sigma X^{\mu\nu}(\sigma) \tilde{h}_{\mu\nu}(x(\sigma)) \\ &= \frac{1}{2} \int d^2\sigma X^{\mu\nu}(\sigma) \int d^4x \tilde{h}_{\mu\nu}(x) \delta^4(x - x(\sigma)) \\ &= \int d^4x \tilde{h}_{\mu\nu}(x) \Theta_{\mu\nu}(x) \end{aligned} \quad (2.112)$$

$$= \int d^4x h_{\mu\nu}(x) \tilde{\Theta}_{\mu\nu}(x), \quad (2.113)$$

where we have introduced the Jacobian,

$$X^{\mu\nu}(\sigma) := \epsilon^{ab} \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} = \frac{\partial(x^\mu, x^\nu)}{\partial(\sigma^1, \sigma^2)}, \quad (2.114)$$

and an antisymmetric tensor of rank two,

$$\Theta_{\mu\nu}(x) := \frac{1}{2} \int d^2\sigma X^{\mu\nu}(\sigma) \delta^4(x - x(\sigma)) = -\Theta_{\nu\mu}(x). \quad (2.115)$$

We call $\Theta_{\mu\nu}(x)$ the vorticity tensor current which has the support on the surface spanned by the Wilson loop C . Note that

$$\Theta^{\mu\nu}(x) = \frac{1}{2} \int_S d^2S^{\mu\nu}(x(\sigma)) \delta^4(x - x(\sigma)), \quad d^2S^{\mu\nu} := d\sigma^1 d\sigma^2 \frac{\partial(x^\mu, x^\nu)}{\partial(\sigma^1, \sigma^2)}. \quad (2.116)$$

Hence we can start from the action,

$$S_M[h^\Lambda; \Theta] = \int d^4x \left\{ \mathcal{L}_m[h^\Lambda] + 2Jg\rho^{-1} K^{1/2} h_{\mu\nu}(x) \tilde{\Theta}_{\mu\nu}(x) \right\}, \quad (2.117)$$

and the expectation value of the Wilson loop is given by

$$\langle W(C) \rangle_{YM} = Z_M[h^\Lambda; \Theta] / Z_M[h^\Lambda; 0], \quad (2.118)$$

where

$$\begin{aligned}
Z_M[h^\Lambda; \Theta] &:= \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \int \mathcal{D}\Lambda_\mu \exp \left\{ i S_M[h^\Lambda; \Theta] \right\} . \\
&= \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[\frac{1}{12\eta^2} (H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4} (\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\} \\
&\quad \times \exp \left\{ i \int d^4x 2Jg\rho^{-1} K^{1/2} h_{\mu\nu}(x) \tilde{\Theta}_{\mu\nu}(x) \right\} \\
&\quad \times \int \mathcal{D}\zeta_\mu \exp \left\{ i \int d^4x \frac{-1}{4} (h_{\mu\nu}^\zeta)^2 \right\} .
\end{aligned} \tag{2.119}$$

Note that the path integral transformation holds,

$$\begin{aligned}
&\int \mathcal{D}\zeta_\mu \exp \left\{ i \int d^4x \frac{-1}{4} (h_{\mu\nu}^\zeta)^2 \right\} \\
&= \int \mathcal{D}\ell_{\mu\nu} \delta(\epsilon^{\mu\nu\rho\sigma} \partial_\rho (\ell_{\mu\nu} - h_{\mu\nu})) \exp \left\{ i \int d^4x \frac{-1}{4} (\ell_{\mu\nu})^2 \right\} ,
\end{aligned} \tag{2.120}$$

where the constraint,

$$\epsilon^{\mu\nu\rho\sigma} \partial_\rho (\ell_{\mu\nu} - h_{\mu\nu}) = 0, \tag{2.121}$$

is solved by

$$\ell_{\mu\nu} - h_{\mu\nu} = \partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu, \quad i.e., \quad \ell_{\mu\nu} = h_{\mu\nu}^\zeta. \tag{2.122}$$

Moreover, we introduce the auxiliary (Abelian) vector field b_μ by

$$\delta(\epsilon^{\mu\nu\rho\sigma} \partial_\rho (\ell_{\mu\nu} - h_{\mu\nu})) = \int \mathcal{D}b_\mu \exp \left\{ -i \int d^4x * b_{\mu\nu} (\ell_{\mu\nu} - h_{\mu\nu}) \right\}, \tag{2.123}$$

where $b_{\mu\nu}$ is the (dual) field strength defined by

$$b_{\mu\nu} := \partial_\mu b_\nu - \partial_\nu b_\mu. \tag{2.124}$$

By using the identity (2.123), the integration over $\ell_{\mu\nu}$ in (2.120) can be performed as

$$\begin{aligned}
&(2.120) \\
&= \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x * b_{\mu\nu} (h_{\mu\nu}) \right\} \int \mathcal{D}\ell_{\mu\nu} \exp \left\{ i \int d^4x \left[\frac{-1}{4} (\ell_{\mu\nu})^2 - * b_{\mu\nu} \ell_{\mu\nu} \right] \right\} \\
&= \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x \left[-(b_{\mu\nu})^2 + * b_{\mu\nu} (h_{\mu\nu}) \right] \right\} .
\end{aligned} \tag{2.125}$$

Another way of deriving the equality just derived, i.e.,

$$\int \mathcal{D}\zeta_\mu \exp \left\{ i \int d^4x \frac{-1}{4} (h_{\mu\nu}^\zeta)^2 \right\} = \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x \left[-(b_{\mu\nu})^2 + * b_{\mu\nu} (h_{\mu\nu}) \right] \right\} \tag{2.126}$$

is as follows. The argument of the exponential in the LHS is

$$(h + d\zeta, h + d\zeta) = (h, h) + (h, d\zeta) + (d\zeta, h) + (d\zeta, d\zeta) \sim (h, h) + (d\zeta, d\zeta), \tag{2.127}$$

under the condition $\delta h = 0$. The last term decouples after the Gaussian integration of ζ . On the other hand, the argument of the the exponential in the RHS is cast into

$$\int d^4x \left[(b_{\mu\nu})^2 - *b_{\mu\nu}(h_{\mu\nu}) \right] = (db, db) - (*db, h) = (b, \delta db) - (b, *dh). \quad (2.128)$$

Suppose the Lorentz type gauge condition $\delta b = 0$. We introduce the NL (zero-form) field ϕ . Then the Gaussian integration over b_μ field yields

$$\begin{aligned} (b, \Delta b) - (b, *dh) - (\delta b, \phi) &= (b, \Delta b) - (b, *dh + d\phi) \\ \rightarrow (*dh + d\phi, \frac{1}{\Delta} *dh + d\phi) &= (h, \frac{\delta d}{\Delta} h) + (\phi, \frac{\delta d}{\Delta} \phi) \sim (h, h) + (\phi, \phi), \end{aligned} \quad (2.129)$$

under the condition $\delta h = 0$. In this derivation, we must insert the constraint $\delta(\partial^\mu b_\mu)$ in the measure $\mathcal{D}b_\mu$. The identity implies that there are many ways of extracting the transverse modes of h .

Thus, the theory is rewritten in terms of b_μ and $h_{\mu\nu}$ as

$$\begin{aligned} &Z_M[b, h; \Theta] \\ &= \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x \left[-(b_{\mu\nu})^2 \right] \right\} \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \\ &\quad \times \exp \left\{ i \int d^4x \left[*b_{\mu\nu} h_{\mu\nu} + \frac{1}{12\eta^2} (H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4} (\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\} \\ &\quad \times \exp \left\{ i \int d^4x 2Jg\rho^{-1} K^{1/2} h_{\mu\nu}(x) \tilde{\Theta}_{\mu\nu}(x) \right\}. \end{aligned} \quad (2.130)$$

By change of variable $b_{\mu\nu} \rightarrow b_{\mu\nu} - 2\rho^{-1} K^{1/2} Jg * \tilde{\Theta}_{\mu\nu}$, we arrive at the expression,

$$\begin{aligned} &Z_M[b, h; \Theta] \\ &= \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x \left[-(b_{\mu\nu} - 2\rho^{-1} K^{1/2} Jg * \tilde{\Theta}_{\mu\nu})^2 \right] \right\} \\ &\quad \times \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[*b_{\mu\nu} h_{\mu\nu} + \frac{1}{12\eta^2} (H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4} (\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\}. \end{aligned} \quad (2.131)$$

2.11 Step 10: Dual Ginzburg-Landau theory in the London limit

First of all, we examine a special case $\gamma = \infty$ for simplicity, although our derivation suggest $\gamma < \infty$. So the last term in (2.131) is neglected. The case of a finite γ is treated in the next section. We change the variable $h_{\mu\nu}$ into the new variable V_μ as follows.¹⁵

$$\int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[\frac{1}{12\eta^2} (H_{\lambda\mu\nu})^2 + *b_{\mu\nu} h_{\mu\nu} \right] \right\}$$

¹⁵ From $\delta h^{(2)} = 0$, there exists a three-form $Y^{(3)}$ such that $h^{(2)} = \delta Y^{(3)} = \delta * W^{(1)} = *dW^{(1)}$. Then $H^{(3)} := dh^{(2)} = d * dW^{(1)} = * \delta dW^{(1)} = *V^{(1)}$, or $V^{(1)} = *H^{(3)}$. Therefore, $\delta V^{(1)} = \delta * H^{(3)} = *dH^{(3)} = *ddh^{(2)} = 0$.

$$= \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[\frac{1}{12\eta^2} (H_{\lambda\mu\nu})^2 + \epsilon^{\mu\nu\rho\sigma} b_\mu \partial_\nu h_{\rho\sigma} \right] \right\} \quad (2.132)$$

$$= \int \mathcal{D}V_\mu \delta(\partial_\mu V^\mu) \exp \left\{ i \int d^4x \left[\frac{-1}{2\eta^2} V_\mu^2 + 2b_\mu V^\mu \right] \right\}, \quad (2.133)$$

since the constraint $\partial_\mu V^\mu = 0$ can be solved by an antisymmetric tensor field in the form,

$$V^\mu := \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu h_{\rho\sigma} = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} = \partial_\nu * h^{\mu\nu}. \quad (2.134)$$

The massive antisymmetric tensor field $h_{\mu\nu}$ denotes the massive spin-1 field V_μ whose canonical mass dimensions is three.¹⁶ In this step, the number of independent degrees of freedom is conserved, since V_μ and $h_{\mu\nu}$ have three independent components.

Furthermore, the integration over V_μ is performed after introducing the new variable θ to remove the delta function of the constraint $\partial_\mu V^\mu = 0$,¹⁷

$$(2.133) = \int \mathcal{D}V_\mu \int \mathcal{D}\theta \exp \left\{ \int d^4x \left[\frac{-i}{2\eta^2} V_\mu^2 + 2iV^\mu b_\mu + i\theta \partial_\mu V^\mu \right] \right\} \quad (2.135)$$

$$= \int \mathcal{D}V_\mu \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[\frac{-1}{2\eta^2} V_\mu^2 + V^\mu (2b_\mu - \partial_\mu \theta) \right] \right\} \quad (2.136)$$

$$= \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[\frac{1}{2} \eta^2 (2b_\mu - \partial_\mu \theta)^2 \right] \right\}. \quad (2.137)$$

Finally, we obtain the dual Abelian gauge theory,

$$\begin{aligned} & Z_M[b, \theta; \Theta] \\ &= \int \mathcal{D}b_\mu \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[-\frac{1}{4} (b_{\mu\nu} + b_{\mu\nu}^S)^2 + \frac{1}{2} \eta^2 (b_\mu - \partial_\mu \theta)^2 \right] \right\}, \end{aligned} \quad (2.138)$$

where

$$b_{\mu\nu}^S(x) := 4\rho^{-1} K^{1/2} Jg * \tilde{\Theta}_{\mu\nu}(x), \quad (2.139)$$

$$\partial^\nu * b_{\mu\nu}^S(x) = \rho^{-1} K^{1/2} \mathcal{D}[\partial] J_\mu^S, \quad J_\mu^S := 4Jg \int_0^1 d\tau \frac{dx_\mu(\tau)}{d\tau} \delta^4(x - x(\tau)), \quad (2.140)$$

and g is the Yang-Mills coupling constant of the original Yang-Mills theory.

This model has dual $U(1)$ symmetry, say $U(1)_m$ symmetry,

$$b_\mu \rightarrow b_\mu + \partial_\mu \vartheta, \quad \theta \rightarrow \theta + \vartheta. \quad (2.141)$$

This model is identified with the London limit $\lambda \rightarrow \infty$ of the dual Abelian Higgs model or the dual Ginzburg-Landau theory with the Lagrangian,

$$\mathcal{L}_{DGL}[b, \phi] = \frac{-1}{4} (b_{\mu\nu} + b_{\mu\nu}^S)^2 + |(\partial_\mu - ig_m b_\mu) \phi|^2 - \lambda(|\phi|^2 - v^2)^2 \quad (2.142)$$

¹⁶The massless antisymmetric tensor field stands for the massless spin-0 field, see [38, 39, 40].

¹⁷In view of the definition (2.134), the V_μ plays the similar role to the magnetic monopole current k_μ defined in the next section. Here the variable θ is introduced to keep the constraint $\partial^\mu k_\mu = 0$. In other words, the introduction of θ keeps the magnetic $U(1)_m$ symmetry. So, putting $\theta = 0$ breaks the $U(1)_m$ symmetry.

where g_m is the magnetic charge subject to the Dirac quantization condition,

$$g_m g = 4\pi. \quad (2.143)$$

The London limit is equivalent to putting $|\phi(x)| = v = \text{const.}$, i.e., $\phi(x) = v \exp[i\theta(x)]$. The dual U(1) symmetry is broken in the London limit,

$$\mathcal{L}_{DGL}[b] = \frac{-1}{4}(b_{\mu\nu} + b_{\mu\nu}^S)^2 + \frac{1}{2}m_b^2 b_\mu b^\mu. \quad (2.144)$$

This corresponds to the infinitesimally thin flux tube connecting the quark and anti-quark. In the London limit, the Higgs mass $m_\phi = 2\sqrt{\lambda}v$ diverges, i.e., $m_\phi = \infty$ or $m_\phi^{-1} = 0$. This is the extreme case of the type II superconductor where $m_\phi > m_b$. It turns out that the mass m_b of the dual gauge field b_μ is given by η ,

$$\eta = m_b = \sqrt{2}g_m v \equiv \frac{\sqrt{2}4\pi}{g}v. \quad (2.145)$$

The monopole condensation is shown to occur in section 5.

3 Final step: Dual Ginzburg-Landau theory of the general type

In this section, we discuss how the LEET given by (2.131) is related to the dual Ginzburg-Landau theory of type II. In section 5, we give another evidence of equivalence between the LEET (2.131) and the dual Ginzburg-Landau theory on the border between type II and type I.

3.1 Dual gauge theory

We return to eq.(2.131):

$$\begin{aligned} & Z_M[b, h; \Theta] \\ &= \int \mathcal{D}b_\mu \exp \left\{ i \int d^4x \left[-(b_{\mu\nu} - 2\rho^{-1}K^{1/2}Jg * \Theta_{\mu\nu})^2 \right] \right\} \\ & \quad \times \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[*b_{\mu\nu}h_{\mu\nu} + \frac{1}{12\eta^2}(H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4}(\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\}. \end{aligned}$$

By making use of the change of variable (2.134), we change the variable $h_{\mu\nu}$ into the new variable V_μ ,

$$\begin{aligned} & \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[*b_{\mu\nu}h_{\mu\nu} + \frac{1}{12\eta^2}(H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4}(\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\} \\ &= \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \left\{ i \int d^4x \left[\epsilon^{\mu\nu\rho\sigma} b_\mu \partial_\nu h_{\rho\sigma} + \frac{1}{12\eta^2}(H_{\lambda\mu\nu})^2 + \frac{1}{4\gamma^4}(\partial^\lambda H_{\lambda\mu\nu})^2 \right] \right\} \\ &= \int \mathcal{D}V_\mu \delta(\partial_\mu V^\mu) \exp \left\{ i \int d^4x \left[2b_\mu V^\mu - \frac{1}{2\eta^2}V_\mu^2 - \frac{1}{4\gamma^4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 \right] \right\}. \quad (3.1) \end{aligned}$$

After introducing the Lagrange multiplier field θ for the constraint $\partial_\mu V^\mu = 0$, the V_μ integration is performed as

$$\begin{aligned}
& (3.1) \\
& = \int \mathcal{D}V_\mu \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[(2b_\mu - \partial_\mu \theta) V^\mu - \frac{1}{2\eta^2} V_\mu^2 - \frac{1}{4\gamma^4} (\partial_\mu V_\nu - \partial_\nu V_\mu)^2 \right] \right\} \\
& = \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[\frac{1}{2} (2b_\mu - \partial_\mu \theta) \frac{\gamma^4}{\Delta - \gamma^4/\eta^2} \left(g^{\mu\nu} - \frac{\eta^2}{\gamma^4} \partial^\mu \partial^\nu \right) (2b_\nu - \partial_\nu \theta) \right] \right\}.
\end{aligned} \tag{3.2}$$

Then we obtain

$$\begin{aligned}
& Z_M[b, \theta; \Theta] \\
& = \int \mathcal{D}b_\mu \delta(\partial^\mu b_\mu) \int \mathcal{D}\theta \exp \left\{ i \int d^4x \left[-\frac{1}{4} (b_{\mu\nu} + b_{\mu\nu}^S)^2 \right. \right. \\
& \quad \left. \left. + \frac{1}{2} (b_\mu - \partial_\mu \theta) \frac{\gamma^4}{\Delta - \gamma^4/\eta^2} \left(g^{\mu\nu} - \frac{\eta^2}{\gamma^4} \partial^\mu \partial^\nu \right) (b_\nu - \partial_\nu \theta) \right] \right\},
\end{aligned} \tag{3.3}$$

where we have inserted the delta function $\delta(\partial^\mu b_\mu)$ for fixing the gauge for b_μ . Note that (3.3) reproduces the London limit when $\gamma \rightarrow \infty$. In the case of finite γ ,

$$\frac{1}{2} (b_\mu - \partial_\mu \theta) \eta^2 \left(1 - \frac{\eta^2}{\gamma^4} \Delta \right)^{-1} \left(g^{\mu\nu} - \frac{\eta^2}{\gamma^4} \partial^\mu \partial^\nu \right) (b_\nu - \partial_\nu \theta) \tag{3.4}$$

$$\sim \frac{1}{2} b_\mu \eta^2 \left(1 + \frac{\eta^2}{\gamma^4} \Delta + \dots \right) b^\mu - \frac{1}{2} \eta^2 \theta \Delta \theta + O(\Delta^2) \tag{3.5}$$

$$= \frac{1}{2} \eta^2 b_\mu b^\mu + \frac{1}{2} \frac{\eta^4}{\gamma^4} b_\mu \Delta b^\mu - \frac{1}{2} \eta^2 \theta \Delta \theta + O(\Delta^2), \tag{3.6}$$

where we have used $\partial^\mu b_\mu = 0$. Thus we arrive at a LEET of the Yang-Mills theory,

$$Z_{APEGT}[b, \theta; 0] = \int \mathcal{D}b_\mu \delta(\partial^\mu b_\mu) \exp \left\{ i \int d^4x \mathcal{L}_K[b, \theta] \right\}, \tag{3.7}$$

$$\mathcal{L}_K[b, \theta] := -\frac{1}{4} \left(1 + \frac{\eta^4}{\gamma^4} \right) (b_{\mu\nu})^2 + \frac{1}{2} \eta^2 b_\mu b^\mu - \frac{1}{2} \eta^2 \theta \Delta \theta. \tag{3.8}$$

The LEET just obtained is of the same form as (2.138), except for the renormalization of the kinetic term of the dual gauge field. The dual gauge field becomes massive, whereas the θ field remains massless. This is reasonable, since the field θ corresponds to the Nambu-Goldstone (NG) mode associated with the spontaneous breakdown of the magnetic U(1) symmetry.

3.2 Low-energy effective theory of dual Abelian Higgs model

In the following we discuss how this theory (3.8) is related to the dual Ginzburg-Landau theory of type II. We remember that the London limit $m_H \rightarrow \infty$ corresponds

to the limit $\gamma = \infty$. Therefore, we expect that the LEET with the Lagrangian (3.8) can be reproduced from the dual Abelian Higgs model, in the region $0 = m_\theta \ll m_b \ll m_H$.

We show that the dual Abelian Higgs model with the Lagrangian,

$$\mathcal{L}_{DGL}[b, \phi] = \frac{-1}{4}(b_{\mu\nu} + b_{\mu\nu}^S)^2 + |(\partial_\mu - ig_m b_\mu)\phi|^2 - \lambda(|\phi|^2 - v^2)^2, \quad (3.9)$$

reduces to the LEET with the Lagrangian (3.8) in the low-energy region. We adopt the renormalizable gauge [41, 42],

$$\partial^\mu b_\mu + \xi m_b \varphi_2 = 0, \quad (3.10)$$

where the scalar field ϕ is parameterized as

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi_1(x) + i\varphi_2(x)]. \quad (3.11)$$

The GF+FP term of the renormalizable gauge is given by

$$\mathcal{L}_{GF+FP} = -\frac{1}{2\xi}(\partial^\mu b_\mu + \xi m_b \varphi_2)^2 + i\bar{c}(\partial^2 + \xi m_b^2)c + ig_m \xi m_b \bar{c}c\varphi_1. \quad (3.12)$$

Even in the Abelian gauge theory, the renormalizable gauge requires the FP ghost and anti-ghost which have a non-trivial interaction with the Higgs scalar φ_1 . A merit of the renormalizable gauge is that the mixing term $m_b b_\mu \partial^\mu \varphi_2$ between b_μ and φ_2 in \mathcal{L}_{GF+FP} cancels the same term in the original Lagrangian,

$$|D_\mu[b]\phi|^2 = |(\partial_\mu - ig_m b_\mu)\phi|^2 \quad (3.13)$$

$$\begin{aligned} &= \frac{1}{2}(\partial_\mu \varphi_1 + g_m b_\mu \varphi_2)^2 + \frac{1}{2}(\partial_\mu \varphi_2 - g_m b_\mu \varphi_1)^2 \\ &\quad - g_m v b^\mu (\partial_\mu \varphi_1 + g_m b_\mu \varphi_1) + \frac{g^2 v^2}{2} b_\mu b^\mu. \end{aligned} \quad (3.14)$$

Note that φ_2 is the would-be Nambu-Goldstone (NG) boson which corresponds to the phase factor θ in the polar coordinate,

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \rho(x)]e^{i\theta(x)}. \quad (3.15)$$

However, we don't use this parameterization in this section, since it is not suitable for performing the loop calculation. Then the total Lagrangian of the DGL theory

$$\mathcal{L}_{DGL}[b, \phi, c, \bar{c}] = \frac{-1}{4}(b_{\mu\nu})^2 + |(\partial_\mu - ig_m b_\mu)\phi|^2 - \lambda(|\phi|^2 - v^2)^2 + \mathcal{L}_{GF+FP}, \quad (3.16)$$

is decomposed into the free part and the interaction part as

$$\begin{aligned} \mathcal{L}_0 &= \frac{1}{2}[(\partial_\mu \varphi_1)^2 - m_{\varphi_1}^2 \varphi_1^2] + \frac{1}{2}[(\partial_\mu \varphi_2)^2 - m_{\varphi_2}^2 \varphi_2^2] \\ &\quad - \frac{1}{4}(\partial_\mu b_\nu - \partial_\nu b_\mu)^2 + \frac{1}{2}m_b^2 b_\mu b^\mu - \frac{1}{2\xi}(\partial^\mu b_\mu)^2 + i\bar{c}(\partial^2 + \xi m_b^2)c, \end{aligned} \quad (3.17)$$

$$\begin{aligned} \mathcal{L}_1 &= g_m b_\mu (\partial^\mu \varphi_1 \varphi_2 - \partial^\mu \varphi_2 \varphi_1) + g^2 v b_\mu b^\mu \varphi_1 + \frac{1}{2}g_m^2 b_\mu b^\mu (\varphi_1^2 + \varphi_2^2) \\ &\quad - \lambda v \varphi_1 (\varphi_1^2 + \varphi_2^2) - \frac{\lambda}{4}(\varphi_1^2 + \varphi_2^2)^2 + ig_m \xi m_b \bar{c}c\varphi_1, \end{aligned} \quad (3.18)$$

where the dual gauge boson mass m_b , the Higgs scalar mass m_{φ_1} and the would-be Nambu-Goldstone mass m_{φ_2} are given by

$$m_b := g_m v, \quad m_{\varphi_1}^2 = 2\mu^2 = \lambda v^2, \quad m_{\varphi_2} := \xi m_b^2. \quad (3.19)$$

The Feynman rules is given in Fig. 3. The propagators are given as follows.

Propagators:

(a) Higgs scalar φ_1 propagator:

$$iD_1(k) = \frac{i}{k^2 - m_{\varphi_1}^2 + i\epsilon}, \quad (m_{\varphi_1}^2 = 2\mu^2 = \lambda v^2). \quad (3.20)$$

(b) Would-be Nambu-Goldstone boson φ_2 propagator:

$$iD_2(k) = \frac{i}{k^2 - m_{\varphi_2}^2 + i\epsilon}, \quad (m_{\varphi_2}^2 = \xi m_b^2 = \xi g_m^2 v^2). \quad (3.21)$$

(c) Gauge boson b_μ propagator:

$$iD_{\mu\nu}(k) = \frac{-i}{k^2 - m_b^2 + i\epsilon} \left[g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2 - \xi m_b^2 + i\epsilon} \right]. \quad (3.22)$$

(d) Ghost (anti-ghost) c (\bar{c}) propagator:

$$iD_c(k) = \frac{i}{k^2 - \xi m_b^2 + i\epsilon}. \quad (3.23)$$

The relevant vertices in the Feynmann rules are given in (e) to (j) of Fig.3.

The basic strategy of showing the equivalence is to integrate out the massive scalar (dual Higgs) field φ_1 with mass squared $m_{\varphi_1}^2 = 2\mu^2 = \lambda v^2$, since the theory (3.3) is written in terms of b_μ and θ ($\theta \sim \varphi_2/\eta$). The renormalized mass $m_H = m_{\varphi_2}$ of the heavy field is made large while all other parameters are held finite. The m_b and m_{φ_1} are the masses of the light fields. The decoupling theorem [18] asserts that phenomena on energy scales much less than the Higgs mass $m_H = m_{\varphi_2}$ are described by a low-energy effective theory with the Lagrangian

$$\begin{aligned} \mathcal{L}_b &= -\frac{1}{4}(\partial_\mu b_\nu - \partial_\nu b_\mu)^2 + \frac{1}{2}m_b^2 b_\mu b^\mu \\ &= -\frac{1}{4}Z_b(\partial_\mu b_\nu^R - \partial_\nu b_\mu^R)^2 + \frac{1}{2}m_b^2 Z_b b_\mu^R b^{\mu R}, \end{aligned} \quad (3.24)$$

where we have substituted the renormalization relation, $b_\mu := Z_b^{1/2} b_\mu^R$. On the other hand, the renormalized Lagrangian with the counter term is given by

$$\mathcal{L}_b = -\frac{1}{4}(\partial_\mu b_\nu^R - \partial_\nu b_\mu^R)^2 + \frac{1}{2}(m_b^R)^2 b_\mu^R b^{\mu R} - \frac{1}{4}\delta_b(\partial_\mu b_\nu^R - \partial_\nu b_\mu^R)^2 + \frac{1}{2}\delta_m b_\mu^R b^{\mu R}. \quad (3.25)$$

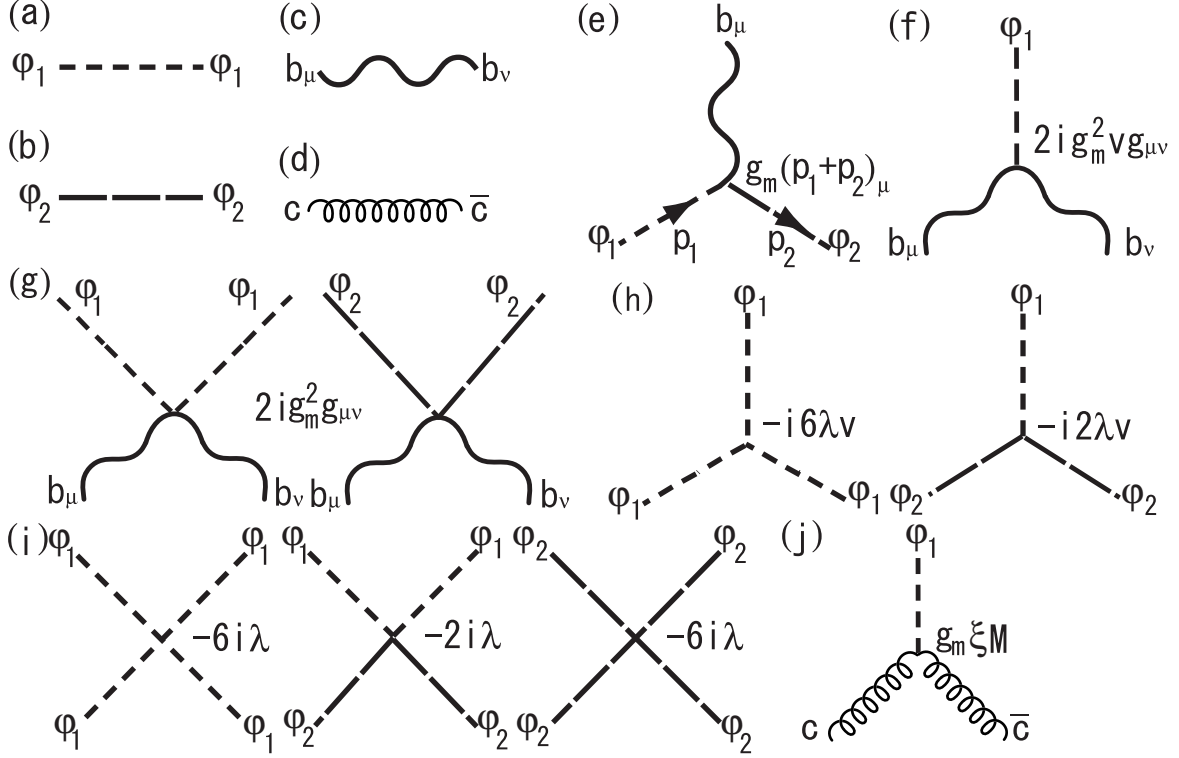


Figure 3: Feynman rules for the dual Ginzburg-Landau model: Propagators: (a) Higgs scalar φ_1 propagator, (b) Would-be Nambu-Goldstone boson φ_2 propagator: (c) Gauge boson b_μ propagator, (d) ghost C propagator; Vertices: (e) $\varphi_1 - \varphi_2 - b$, (f) $\varphi_1 - b - b$, (g) $\varphi_1 - \varphi_1 - b - b$, $\varphi_2 - \varphi_2 - b - b$, (h) $\varphi_1 - \varphi_1 - \varphi_1$, $\varphi_1 - \varphi_2 - \varphi_2$, (i) $\varphi_1 - \varphi_1 - \varphi_1 - \varphi_1$, $\varphi_1 - \varphi_1 - \varphi_2 - \varphi_2$, $\varphi_2 - \varphi_2 - \varphi_2 - \varphi_2$, (j) $\varphi_1 - c - \bar{c}$.

Equating (3.24) and (3.25), we obtain the relationship,

$$\delta_b := Z_b - 1, \quad \delta_m := Z_b m_b^2 - (m_b^R)^2. \quad (3.26)$$

We use the renormalized perturbation theory in the (dual) coupling constant

$$g_m := \frac{4\pi}{g}. \quad (3.27)$$

In order to see coincidence of the resultant theory with Lagrangian (3.8), it is useful to choose the Landau gauge $\xi = 0$ in which the would-be NG boson φ_2 is massless, since $m_{\varphi_2}^2 = \xi m_b^2$. In Appendix E, we evaluate one-loop graphs for the self-energy of the would-be NG boson φ_2 and the vacuum polarization of the dual gauge boson b_μ as shown in Fig. 5 and Fig.6 respectively, based on the Feynman rules given in Fig. 3. The radiative correction causes the mass renormalization and wavefunction renormalization of b_μ field, while the would-be NG boson field φ_2 field remains massless for $\xi = 0$.

Now the renormalized mass is written as

$$(m_b^R)^2 = Z_b m_b^2 - \delta_m = m_b^2 + m_b^2 \delta_b - \delta_m, \quad (3.28)$$

where $(m_b^R)^2$ is of order $g_m^2 m_b^2$, whereas δ_m is of order $g_m^2 m_b^2$ or $g_m^2 m_H^2$. Actual calculations in Appendix E give

$$\begin{aligned}\delta_b &= -\frac{g_m^2}{(4\pi)^2} \left(\frac{N_\epsilon}{3} + \frac{13}{36} + 6 \ln \frac{m^2}{\mu^2} \right) + O(g_m^4), \\ \delta_m &= \frac{4g_m^2}{(4\pi)^2} m_b^2 \left[\frac{3}{4} (N_\epsilon + 1) - \frac{m_H^2 \ln \frac{m_H^2}{\mu^2} - m_b^2 \ln \frac{m_b^2}{\mu^2}}{m_H^2 - m_b^2} - \frac{1}{8} \frac{m_H^2}{m_b^2} \left(-1 + 2 \ln \frac{m_H^2}{\mu^2} \right) \right] \\ &\quad - \frac{g_m^2}{(4\pi)^2} m_H^2 \left(-N_\epsilon + \frac{1}{2} + \ln \frac{m_H^2}{\mu^2} \right) + O(g_m^4),\end{aligned}\tag{3.29}$$

where

$$N_\epsilon := \frac{1}{\epsilon} + \ln 4\pi - \gamma_E.\tag{3.30}$$

Substituting (3.29) into (3.28), we obtain

$$\begin{aligned}(m_b^R)^2 &= m_b^2 - \frac{g_m^2}{(4\pi)^2} m_H^2 \left(\frac{4}{3} N_\epsilon + \frac{49}{36} + 5 \ln \frac{m_H^2}{\mu^2} - \ln \frac{m_b^2}{\mu^2} \right) \\ &\quad - \frac{4g_m^2}{(4\pi)^2} m_b^2 \left[\frac{3}{4} (N_\epsilon + 1) - \frac{m_H^2 \ln \frac{m_H^2}{\mu^2} - m_b^2 \ln \frac{m_b^2}{\mu^2}}{m_H^2 - m_b^2} \right] + O(g_m^4).\end{aligned}\tag{3.31}$$

In order to keep the physical mass m_b^R of b_μ finite even for the large dual Higgs mass, $m_H \rightarrow \infty$, we must let m_b^2 have a term proportional to $g_m^2 m_H^2$,

$$m_b^2 = \text{finite term} + \frac{g_m^2}{(4\pi)^2} m_H^2 \left(\frac{4}{3} N_\epsilon + \frac{49}{36} + 5 \ln \frac{m_H^2}{\mu^2} - \ln \frac{m_b^2}{\mu^2} \right).\tag{3.32}$$

Thus we arrive at the conclusion,

$$\begin{aligned}(m_b^R)^2 &= \text{finite term in } m_b \\ &\quad - \frac{4g_m^2}{(4\pi)^2} m_b^2 \left[\frac{3}{4} (N_\epsilon + 1) - \frac{m_H^2 \ln \frac{m_H^2}{\mu^2} - m_b^2 \ln \frac{m_b^2}{\mu^2}}{m_H^2 - m_b^2} \right] + O(g_m^4).\end{aligned}\tag{3.33}$$

The comparison with (3.8) implies that

$$Z_b = 1 + \frac{\eta^4}{\gamma^4}, \quad (m_b)^2 Z_b = \eta^2.\tag{3.34}$$

3.3 Comparison with the previous work

In the previous paper [11], we have used the Hodge decomposition for the two-form B ,

$$B = db + *d\chi,\tag{3.35}$$

with two one-forms b and χ . Then the definition of h yields

$$h := *B = *db + **d\chi = \delta * b - d\chi,\tag{3.36}$$

which leads to

$$\delta h = -\delta d\chi, \quad (3.37)$$

$$H := dh = d\delta * b = *\delta db, \quad (3.38)$$

where we have used $dd = 0 = \delta\delta$. Moreover,

$$\delta H = \delta dh = \delta * \delta db = *\delta\delta db = *d(\delta d + d\delta)b = *d\Delta b. \quad (3.39)$$

Hence, if we require $\delta h = 0$, then $\Delta\chi = 0$ under the gauge fixing condition $\delta\chi = 0$. It is known [34] that a p -form ω is harmonic ($\Delta\omega = 0$) if and only if ω is closed ($d\omega = 0$) and co-closed ($\delta\omega = 0$). The harmonic form ω does not exist on the topologically trivial manifold, since the dimension of the set of exact p -forms is equal to the Betti number, i.e., $\dim \text{Harm}^p(M) = b^p$. In this case, χ is divergenceless and rotation free vector field on the four-dimensional Minkowski spacetime. Hence we can eliminate the variable χ and hence χ does not appear in the result. This corresponds to the situation discussed in the paper [11].

It is instructive to compare the above result with the previous one. Replacing $V^\mu = \partial_\nu * h^{\mu\nu}$ with the magnetic monopole current k_μ (this definition is suggested from $h \sim *B$ since $k \sim \delta B$), we obtain the theory,

$$\begin{aligned} Z_{APEGT}[b, k; \Theta] &= \int \mathcal{D}b_\mu \int \mathcal{D}k_\mu \exp \left\{ i \int d^4x \left[-\frac{1}{4}(b_{\mu\nu} + b_{\mu\nu}^S)^2 \right. \right. \\ &\quad \left. \left. + b_\mu k^\mu + \frac{1}{2\eta^2} k_\mu^2 + \frac{1}{2\gamma^4} (k, \Delta k) \right] \right\}. \end{aligned} \quad (3.40)$$

After integrating out the dual gauge field b_μ , this action leads to the theory of magnetic monopole written in terms only of the monopole current k_μ . See section 5.

3.4 How to determine the parameters ρ, σ, α

It has been shown [17, 11, 35] that the renormalization of the Yang-Mills coupling constant g does not depend on ρ, σ and the gauge parameter α . Therefore, the β -function is also independent of the choice of ρ, σ and of the gauge parameter α . The resulting β -function exactly coincides with the one-loop β -function of the original $SU(N)$ Yang-Mills theory,

$$\beta(g_R) := \mu \frac{\partial g_R}{\partial \mu} = -g_R \mu \frac{\partial}{\partial \mu} \ln Z_g = -\frac{b_0}{(4\pi)^2} g_R^3 + O(g_R^5), \quad b_0 = \frac{11}{3} C_2(G), \quad (3.41)$$

exhibiting the asymptotic freedom. Moreover, some of the anomalous dimensions have been calculated.¹⁸ The anomalous dimensions of the fields in the $SU(N)$ Yang-Mills

¹⁸The anomalous dimensions of the fields a_μ^i , $B_{\mu\nu}^i$ and the parameters ρ, β are calculated in the previous paper [35]. To obtain the anomalous dimension for σ, α , we need to calculate more Feynman graphs than those in [35], see [72]. For $G = SU(2)$, Z_α and Z_ζ were calculated by Hata and Niigata [32] and Schaden [19], Z_A by Schaden [19] and Z_{C^a} by Quandt and Reinhardt [17].

theory are evaluated as

$$\begin{aligned}
\gamma_{a^i}(g) &:= \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_a = \frac{11}{3}C_2(G)\frac{g_R^2}{(4\pi)^2} \quad [SU(N)], \\
\gamma_{B^i}(g) &:= \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_B = -\frac{1+\alpha_R}{2}\sigma_R^2 C_2(G)\frac{g_R^2}{(4\pi)^2} \quad [SU(N)], \\
\gamma_{A^a}(g) &:= \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_A = \left(\frac{22}{3} - \frac{9}{2} - \frac{\alpha_R}{2} - \beta_R\right) C_2(G)\frac{g_R^2}{(4\pi)^2} \quad [SU(2)], \\
\gamma_{C^a}(g) &:= \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_{C^a} = (\beta_R - 3)\frac{g_R^2}{(4\pi)^2} \quad [SU(2)], \\
\gamma_{C^i}(g) &:= \frac{1}{2}\mu \frac{\partial}{\partial \mu} \ln Z_{C^i} = ?, \tag{3.42}
\end{aligned}$$

where ? denotes that the result is not yet unavailable. For the parameters $\rho, \sigma, \alpha, \beta$ and ζ ,

$$\begin{aligned}
\gamma_\rho(g) &:= \mu \frac{\partial \rho_R}{\partial \mu} = -\rho_R \mu \frac{\partial}{\partial \mu} \ln Z_\rho \\
&= -\rho_R \left[-\frac{11}{6} - \frac{\sigma_R^2}{2} + 2\frac{\sigma_R}{\rho_R} - \frac{1-\alpha_R}{2} \left(\frac{\sigma_R}{\rho_R} - \frac{\sigma_R^2}{2} \right) \right] C_2(G)\frac{g_R^2}{(4\pi)^2} \quad [SU(N)], \\
\gamma_\sigma(g) &:= \mu \frac{\partial \sigma_R}{\partial \mu} = -\sigma_R \mu \frac{\partial}{\partial \mu} \ln Z_\sigma = ?, \\
\gamma_\alpha(g) &:= \mu \frac{\partial \alpha_R}{\partial \mu} = -\alpha_R \mu \frac{\partial}{\partial \mu} \ln Z_\alpha = -\alpha_R \left(\frac{3}{\alpha_R} + 6 - \frac{22}{3} + \alpha_R \right) \frac{g_R^2}{8\pi^2} \quad [SU(2)], \\
\gamma_\beta(g) &:= \mu \frac{\partial \beta_R}{\partial \mu} = -\beta_R \mu \frac{\partial}{\partial \mu} \ln Z_{a^i} = -2\gamma_{a^i}(g)\beta_R \quad [SU(N)], \\
\gamma_\zeta(g) &:= \mu \frac{\partial \zeta_R}{\partial \mu} = -\zeta_R \mu \frac{\partial}{\partial \mu} \ln Z_\zeta = \alpha_R(1 + \zeta_R) \left(\zeta_R + \frac{3}{\alpha_R^2} \right) \quad [SU(2)]. \tag{3.43}
\end{aligned}$$

Note that the parameters $\rho, \sigma, \alpha, \beta$ do run and changes according to the scale μ . We would like to obtain the renormalization scale μ independent result. The simplest way is to search for the fixed point for these parameters on which the parameters are kept fixed irrespective of the renormalization scale μ . The fixed point is determined by solving the simultaneous equations, $\gamma_\rho(g) = 0$, $\gamma_\sigma(g) = 0$, $\gamma_\alpha(g) = 0$ and $\gamma_\zeta(g) = 0$.

Finally, we give a simple argument how to determine the parameters ρ, σ, α in our theory. The parameter ρ obeys the differential equation,

$$\gamma_\rho(g) := \mu \frac{\partial \rho_R}{\partial \mu} = \frac{1}{2} \left[\left(\frac{11}{3} + \frac{1+\alpha}{2}\sigma^2 \right) \rho_R - (3 + \alpha)\sigma \right] C_2(G)\frac{g_R^2}{(4\pi)^2}. \tag{3.44}$$

This implies that the point $\rho = \rho^*$,

$$\rho = \rho^* := \frac{3 + \alpha}{\frac{11}{3} + \frac{1+\alpha}{2}\sigma^2}, \tag{3.45}$$

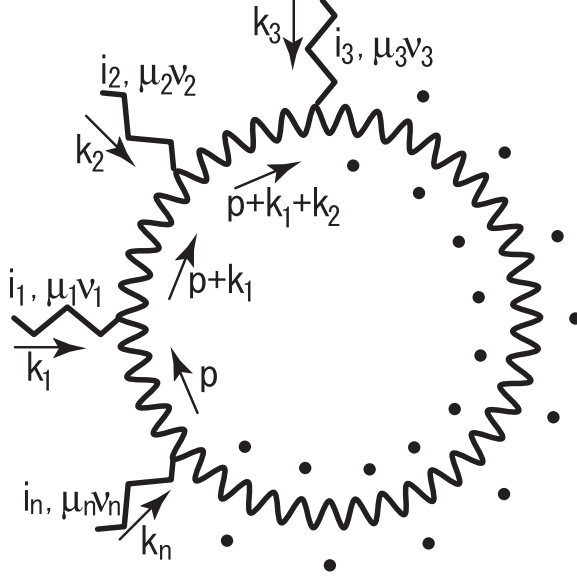


Figure 4: Higher order terms for obtaining corrections of APEGT.

is the infrared fixed point, as far as $\frac{11}{3} + \frac{1+\alpha}{2}\sigma^2 > 0$. At least in the present stage of investigations, the simplest choice of the parameters is

$$\alpha_R = 1, \quad \sigma_R = 1. \quad (3.46)$$

The choice $\alpha = 1$ greatly simplifies the evaluation and the expression of the result for the vacuum polarization of $B_{\mu\nu}$. The choice $\sigma = 1$ completely eliminates the quartic gluon interaction, see (2.45). Usually, the auxiliary field $B_{\mu\nu}$ is introduced so as to achieve this situation. In this case, the value

$$\rho_R = \frac{6}{7} \quad (3.47)$$

is the infrared fixed point. However, (3.43) implies that $\alpha_R(\mu)$ monotonically decreasing in μ and that there is no fixed point for α in the $SU(2)$ case, although the $SU(N)$ case can be different from the $SU(2)$ case.

The complete list of the anomalous dimensions in the $SU(N)$ case and the details of the RG properties of the APEGT will be given in a subsequent paper [72].

4 Estimation of neglected higher-order terms

We proceed to discuss the issue whether the neglected terms in the above derivation do not invalidate the above result in the range of parameters and the energy scale in question.

4.1 Higher-order cumulants and large N suppression

In this paper we have obtained the APEGT which is bilinear with respect to the diagonal fields, a_μ and $B_{\mu\nu}$. Hence the bilocal approximation in section 2.3 is exact within this framework. In the framework of large N expansion, moreover, neglecting higher-order cumulants can be justified as follows.¹⁹ In the large N expansion, $\lambda := g^2 N$ is kept fixed.

For example, we consider the diagram in Fig.4 with n external $B_{\mu\nu}$ lines. According to the Feynman rules in Fig. 1, it corresponds to

$$\begin{aligned}
& \Pi_{\mu_1\nu_1, \dots, \mu_n\nu_n}^{i_1 \dots i_n}(k_1, \dots, k_n) \\
&= \frac{1}{n} \int \frac{d^4 p}{(2\pi)^4} D_{\sigma_n \rho_1}^{d_n c_1}(p) [-2g\sigma f^{i_1 c_1 d_1} I_{\mu_1 \nu_1, \rho_1 \sigma_1}] \\
& \quad \times D_{\sigma_1 \rho_2}^{d_1 c_2}(p + k_1) [-2g\sigma f^{i_2 c_2 d_2} I_{\mu_2 \nu_2, \rho_2 \sigma_2}] \\
& \quad \dots \\
& \quad \times D_{\sigma_{n-1} \rho_n}^{d_{n-1} c_n}(p + k_1 + \dots + k_{n-1}) [-2g\sigma f^{i_n c_n d_n} I_{\mu_n \nu_n, \rho_n \sigma_n}].
\end{aligned} \tag{4.1}$$

This quantity is proportional to the factor,

$$g^n \delta^{d_n c_1} f^{i_1 c_1 d_1} \delta^{d_1 c_2} f^{i_2 c_2 d_2} \dots \delta^{d_{n-1} c_n} f^{i_n c_n d_n}. \tag{4.2}$$

For $n = 2$, the factor reads

$$g^2 \delta^{d_2 c_1} f^{i_1 c_1 d_1} \delta^{d_1 c_2} f^{i_2 c_2 d_2} = g^2 f^{i_1 c_1 c_2} f^{i_2 c_2 c_1} = -g^2 C_2 \delta^{i_1 i_2} = -g^2 N \delta^{i_1 i_2} \sim O(N^0). \tag{4.3}$$

For $n = 3$, it is identically zero,

$$g^3 \delta^{d_3 c_1} f^{i_1 c_1 d_1} \delta^{d_1 c_2} f^{i_2 c_2 d_2} \delta^{d_2 c_3} f^{i_3 c_3 d_3} = g^3 f^{i_1 c_1 c_2} f^{i_2 c_2 c_3} f^{i_3 c_3 c_1} = 0. \tag{4.4}$$

For $n = 4$, it is of order $O(N^{-2})$ for large N , since

$$\begin{aligned}
& g^4 \delta^{d_4 c_1} f^{i_1 c_1 d_1} \delta^{d_1 c_2} f^{i_2 c_2 d_2} \delta^{d_2 c_3} f^{i_3 c_3 d_3} \delta^{d_3 c_4} f^{i_4 c_4 d_4} \\
&= g^4 f^{i_1 c_1 c_2} f^{i_2 c_2 c_3} f^{i_3 c_3 c_4} f^{i_4 c_4 c_1} \\
&= g^4 \frac{N}{N+1} (\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3}) \sim O(N^{-2}).
\end{aligned} \tag{4.5}$$

The last equality is obtained as follows. From the symmetry under the exchange of the indices i_1, i_2, i_3, i_4 , we can put

$$f^{i_1 c_1 c_2} f^{i_2 c_2 c_3} f^{i_3 c_3 c_4} f^{i_4 c_4 c_1} = A(\delta^{i_1 i_2} \delta^{i_3 i_4} + \delta^{i_1 i_3} \delta^{i_2 i_4} + \delta^{i_1 i_4} \delta^{i_2 i_3}). \tag{4.6}$$

By contracting i_3 and i_4 , we obtain

$$LHS = f^{i_1 c_1 c_2} f^{i_2 c_2 c_3} f^{i_3 c_3 c_4} f^{i_3 c_4 c_1} = -f^{i_1 c_1 c_2} f^{i_2 c_2 c_3} \delta^{c_1 c_3} = N \delta^{i_1 i_2}, \tag{4.7}$$

$$RHS = A[\delta^{i_1 i_2} (N-1) + 2\delta^{i_1 i_2}] = A(N+1) \delta^{i_1 i_2}, \tag{4.8}$$

¹⁹ The author would like to thank Toru Shinohara for helpful discussion on this point.

where we have used the formula (A.3). Hence we obtain $A = N/(N + 1)$. This argument can be extended to arbitrary n . Thus it turns out that the higher-order terms with n external lines of the tensor fields B are suppressed by $1/N^2$ in the large N expansion for $n \geq 4$.

Thus the contribution from the diagrams with n external lines of diagonal fields is suppressed by a factor $1/N^2$ for $n \geq 4$ where n is the total number of external diagonal gluon fields a_μ^i and external tensor gauge fields $B_{\mu\nu}^i$ in the off-diagonal one-gluon-loop or one-ghost-loop diagram. This is because every three-point vertex (c),(d),(e) in Feynman rules has a common factor gf^{iab} . In the leading order of the large N expansion, we have only to take into account the diagrams with two external lines and hence the resulting APEGT is bilinear in the diagonal fields, a_μ^i (or $f_{\mu\nu}^i$) and $B_{\mu\nu}^i$. In this limit, therefore, the bilocal approximation of neglecting the higher-order cumulants is exact within our approach. In this sense, the bilocal approximation is consistent within the framework of the APEGT. This situation is in sharp contrast to the bilocal approximation in the analytical treatment of the stochastic vacuum model, although the validity is confirmed by the numerical calculations on a lattice[30].

4.2 Higher-order terms of low-energy or large mass expansion and the decoupling theorem

We have neglected higher-order terms of the large mass or the derivative expansion in powers of p^2/M_A^2 . This approximation will be valid in the low-energy region below M_A . This is considered as an example of the Appelquist-Carazzone decoupling theorem[18].

First, we recall the case of QED. A typical example of applying the decoupling theorem is a derivative expansion of the photon effective action (known as the Euler-Heisenberg Lagrangian) obtained by integrating out the electron field in QED as

$$\begin{aligned} \Gamma_{eff}[a] &= -i \ln \int [d\psi][d\bar{\phi}] \exp \{iS_{QED}\} \end{aligned} \quad (4.9)$$

$$\begin{aligned} &= \frac{-1}{4} \int d^4x f_{\mu\nu} f^{\mu\nu} - \frac{e^2}{3(4\pi)^2} z(\mu) \int d^4x f_{\mu\nu} f^{\mu\nu} - \frac{e^2}{15(4\pi)^2 M^2} \int d^4x f_{\mu\nu} \partial^2 f^{\mu\nu} \\ &+ \frac{e^4}{90(4\pi)^2 M^4} \int d^4x \left[(f_{\mu\nu} f^{\mu\nu})^2 + \frac{7}{4} (f_{\mu\nu} * f^{\mu\nu})^2 \right] + \left(\frac{p^2}{M^2} \right)^3, \end{aligned} \quad (4.10)$$

where M is the electron mass and

$$z(\mu) := \frac{2}{\epsilon} + \ln 4\pi - \gamma_E - \ln \frac{M^2}{\mu^2}. \quad (4.11)$$

It turns out that the terms that do not decouple in the $M \rightarrow \infty$ limit have the same form as those appearing in the original Lagrangian and therefore they can be absorbed in the wavefunction renormalization. The new structures appear as non-renormalizable terms and they vanish in the limit $M \rightarrow \infty$. This is an example of the decoupling theorem [18]. The theorem states that, under some given conditions, the effects of the heavy particle only appear in the light particle physics through

corrections proportional to a negative power of M or through renormalization. The validity of this approach is limited to energies much lower than the mass M .

In the low-energy effective action of the Yang-Mills theory, all the terms including more than two diagonal fields can be suppressed in the large N limit, as we discussed in the above. Of course, this argument does not hold for the relatively small N . Even in the case of not-so-large N (e.g., $N = 2$), however, such terms are suppressed by the power of the inverse (off-diagonal-gluon) mass $1/M_A$. In the limit $M_A \rightarrow \infty$, the off-diagonal gluons affect the physics described by the diagonal gluons only through renormalization. In the case of small N , the validity of our approach is limited to low energies lower than the off-diagonal gluon mass M_A . More quantitative estimate of the higher order terms will be given in a subsequent paper [72].

5 Magnetic monopole condensation and area law

In this section, we show that the LEET given by (2.131) reproduces the same results as those obtained by the supposed dual Ginzburg-Landau theory of type II. Here the dual Ginzburg-Landau theory can include the type II on the border of type I. Therefore, two extreme limits, London limit and Bogomol'nyi limit, are special cases.

5.1 Monopole action and monopole condensation

In order to obtain the monopole action, we return to the action (2.117),

$$S_M[h; \Theta] = \int d^4x \left\{ \mathcal{L}_m[h] + 2Jg\rho^{-1}K^{1/2}h_{\mu\nu}(x)\tilde{\Theta}_{\mu\nu}(x) \right\}, \quad (5.1)$$

where the expectation value of the Wilson loop is evaluated as

$$\langle W(C) \rangle_{YM} = Z_M[h; \Theta] / Z_M[h; 0], \quad (5.2)$$

$$Z_M[h; \Theta] := \int \mathcal{D}h_{\mu\nu} \delta(\partial^\nu h_{\mu\nu}) \exp \{ i S_M[h; \Theta] \}. \quad (5.3)$$

We define the monopole current k_μ by

$$k_\mu := g_m \partial^\nu * h_{\mu\nu} = g_m K^{1/2} \partial^\nu B_{\mu\nu}, \quad (5.4)$$

which satisfies the (topological) conservation law, $\partial^\mu k_\mu = 0$. Conversely, the tensor field h is written in terms of the monopole current k as

$$h_{\mu\nu} = g_m^{-1} \epsilon_{\mu\nu\rho\sigma} \frac{1}{\partial^2} \partial^\rho k^\sigma = g_m^{-1} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \frac{1}{\partial^2} (\partial^\rho k^\sigma - \partial^\sigma k^\rho), \quad (5.5)$$

which is subject to the constraint, $\partial^\mu h_{\mu\nu} = 0$. It is easy to show that

$$\int d^4x \frac{g_m^2}{2} (h_{\mu\nu})^2 = (k, \Delta^{-1}k) + (\delta h, \Delta^{-1}\delta h) \sim (k, \Delta^{-1}k), \quad (5.6)$$

$$- \int d^4x \frac{g_m^2}{6} (H_{\lambda\mu\nu})^2 = (k, k), \quad (5.7)$$

$$\int d^4x \frac{g_m^2}{2} (\partial^\lambda H_{\lambda\mu\nu})^2 = (k, \Delta k), \quad (5.8)$$

where we have defined $\Delta := d\delta + \delta d$ and used $\delta h = 0$. Thus, we obtain the magnetic monopole theory as a LEET of Yang-Mills theory,

$$\boxed{S_{MP}[k] = \int d^4x \frac{1}{g_m^2} \left[\frac{-1}{2}(k, \Delta^{-1}k) - \frac{1}{2\eta^2}(k_\mu)^2 + \frac{1}{2\gamma^4}(k, \Delta k) + O\left(\frac{(k, \Delta^2 k)}{M_A^6}\right) \right]}, \quad (5.9)$$

where η and γ are given by (2.82) and (2.83) respectively. Hence the expectation value of the Wilson loop is obtained from

$$\boxed{\langle W(C) \rangle_{YM} = Z_{MP}[k; \Xi] / Z_{MP}[k; 0],} \quad (5.10)$$

and

$$\boxed{Z_{MP}[k; \Xi] := \int \mathcal{D}k_\mu \exp \left\{ iS_{MP}[k] + i2Jg\rho^{-1}K^{1/2}g_m^{-1} \int d^4x k_\mu \mathcal{D}[\partial]\Xi^\mu \right\},} \quad (5.11)$$

where Ξ_μ denotes the four-dimensional solid angle under which the surface²⁰ S with the two-dimensional coordinate (σ^1, σ^2) is seen by an observer at the point x ,

$$\Xi_\mu(x) := \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \partial_x^\nu \int_S dS^{\rho\sigma}(x(\sigma)) \Delta^{-1}(x - x(\sigma)) \quad (5.12)$$

$$= \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} \partial_x^\nu \int_S dS^{\rho\sigma}(x(\sigma)) \frac{1}{(x - x(\sigma))^2}. \quad (5.13)$$

Here note that the total number of independent degrees of freedom is unchanged under the change of variables from $h_{\mu\nu}$ to k_μ and that the associated Jacobian in the integration measure $\mathcal{D}k_\mu$ is field independent and hence omitted.

From the monopole action, we can demonstrate that the monopole condensation does really occur in the sense $\langle k^\mu k_\mu \rangle \neq 0$. In the London limit $\gamma^{-1} = 0$, the propagator of the monopole current is given by

$$\langle k_\mu(x) k_\nu(y) \rangle = g_{\mu\nu} \int \frac{d^4p}{(2\pi)^{4i}} \left(\frac{1}{p^2} - \frac{1}{\eta^2} \right)^{-1} e^{ip(x-y)} = \eta^2 g_{\mu\nu} \int \frac{d^4p}{(2\pi)^{4i}} \frac{p^2}{\eta^2 - p^2} e^{ip(x-y)}. \quad (5.14)$$

Therefore, we obtain non-vanishing monopole condensation for non-vanishing off-diagonal gluon mass,

$$\langle k_\mu(x) k_\mu(x) \rangle = 4 \int \frac{d^4p}{(2\pi)^4} \frac{\eta^2 p^2}{\eta^2 - p^2} = \frac{1}{4\pi^2} \eta^6 (\ln 4\pi - \gamma_E + 1 - \ln \eta^2), \quad (5.15)$$

where we have used the MS scheme of the dimensional regularization. This should be compared with the mass m_b of the dual gauge field b_μ . An close relationship between the monopole condensation and the mass of the dual gauge field was conjectured in

²⁰An apparent S dependence should drop out after summing over branches of the multivalued potential.

the previous work [11]. In fact, the above propagator for the monopole current leads to

$$\langle k_\mu(x)k_\nu(y) \rangle = -\eta^2 \delta_{\mu\nu} \int \frac{d^4 p}{(2\pi)^4} \left(1 - \frac{\eta^2}{p^2}\right)^{-1} e^{ip(x-y)} = -\eta^2 \delta_{\mu\nu} \delta(x-y) + O(\eta^4), \quad (5.16)$$

which is nothing but eq.(4.22) predicted in the previous paper [11]. In this paper we have shown that *the origin of monopole condensation is the existence of off-diagonal gluon mass M_A which provides also the mass m_b of the dual gauge field b_μ ,*

$$\boxed{\langle k_\mu k_\mu \rangle \neq 0 \leftrightarrow M_A \neq 0 \leftrightarrow m_b \neq 0.} \quad (5.17)$$

The monopole action (5.9) is written in the form,

$$S_{MP}[k] = -\frac{1}{2g_m^2}(\tilde{k}, D_m(p^2)\tilde{k}), \quad (5.18)$$

where the inverse of $D_m(p)$ is the propagator given by

$$D_m^{-1}(p^2) := \left(\frac{1}{p^2} - \frac{1}{\eta^2} + \frac{p^2}{\gamma^4} \right)^{-1} = \chi \left(\frac{p^2}{p^2 - m_1^2} - \frac{p^2}{p^2 - m_2^2} \right), \quad (5.19)$$

where

$$\boxed{m_{1,2}^2 := \frac{\gamma^4}{2\eta^2} \left(1 \pm \sqrt{1 - 4\eta^4/\gamma^4} \right) \quad (m_1 \geq m_2),} \quad (5.20)$$

$$\boxed{\chi := \frac{m_1^2 m_2^2}{m_1^2 - m_2^2} = \frac{\eta^2}{\sqrt{1 - 4\eta^4/\gamma^4}}.} \quad (5.21)$$

Finally, the monopole condensation is calculated according to

$$\langle k_\mu k_\mu \rangle = 4 \int \frac{d^D p}{(4\pi)^D} D_m^{-1}(p), \quad (5.22)$$

which yields

$$\boxed{\langle k_\mu k_\mu \rangle = \frac{\chi}{4\pi^2} \left[(\ln 4\pi - \gamma_E + 1)(m_1^4 - m_2^4) - m_1^4 \ln m_1^2 + m_2^4 \ln m_2^2 \right].} \quad (5.23)$$

Thus we obtain non-zero monopole condensate.

5.2 Area law of the Wilson loop

Thanks to the NAST, the Wilson loop operator has an alternative form expressed in terms of the monopole current. The expectation value is expressed by (5.11). By making use of the monopole action (5.18), the VEV of the large Wilson loop is calculated by performing the Gaussian integration over k_μ as

$$\langle W(C) \rangle_{YM} = \exp \left\{ -\frac{1}{2} (2Jg\rho^{-1}K^{1/2})^2 (\tilde{\Xi}_\mu, D_m^{-1} \tilde{\Xi}^\mu) \right\}. \quad (5.24)$$

It is not difficult to show (see Appendix F) that (5.24) leads to the area law,

$$\boxed{\langle W(C) \rangle_{YM} \cong \exp[-\sigma_{st} A(C)],} \quad (5.25)$$

for the large Wilson loop. The string tension is obtained as

$$\boxed{\sigma_{st} = \frac{J^2 g^2}{2\pi} \rho^{-2} K \chi \ln \frac{m_1^2}{m_2^2}.} \quad (5.26)$$

This is one of main results of this paper. The static potential is defined for the rectangular loop with side lengths T, R as

$$V(R) := -\lim_{T \rightarrow \infty} \frac{1}{T} \ln \langle W(C) \rangle_{YM}. \quad (5.27)$$

Then we obtain the linear static potential,

$$\boxed{V(R) = \sigma_{st} R,} \quad (5.28)$$

for large separation R . This result is consistent with the claim that the QCD vacuum is the dual superconductor (of type II). This result indicates that quark in any representation is confined in the SU(2) case. For SU(3), our result can be applied only to quark in the fundamental representation due to a restriction coming from NAST.

The factor $\rho^{-2}K$ is estimated as follows. In particular, when $\alpha = 1$, f_0 is given by

$$f_0 = -\ln \frac{M_A^2}{\mu^2} - \gamma_E + \ln 4\pi = -\ln \frac{g^2}{16\pi^2} - 1 + \frac{16\pi^2}{b_0 g^2}, \quad (5.29)$$

where we have used the expression of off-diagonal gluon mass (2.40). Then K reads

$$K := 1 + \frac{Ng^2}{2\pi^2} \sigma^2 f_0 = 1 + \frac{24}{11} \sigma^2 - \frac{Ng^2}{2\pi^2} \sigma^2 \left(\ln \frac{g^2}{16\pi^2} + 1 \right) := K(g), \quad (5.30)$$

since $C_2 = N$ and $b_0 = \frac{11}{3}N$ for $G = SU(N)$. For the choice $\sigma = 1$ and $\rho = 6/7$ in (3.47), the factor reads

$$\rho^{-2}K = \frac{49}{36} \left[\frac{35}{11} - \frac{Ng^2}{2\pi^2} \left(\ln \frac{g^2}{16\pi^2} + 1 \right) \right], \quad (5.31)$$

which is positive for $0 \leq g < 9.6$ and monotonically increases from $[4.3, 6.5]$ for $g \in [0, 4.6]$ with a peak at $g \cong 4.6$ and monotonically decreases with $[6.5, 0]$ for $g \in [4.6, 9.6]$. Therefore, we can put $\rho^{-2}K \cong 5$ for $g \cong 2$ in the SU(2) case. The numerical estimation of χ, m_1, m_2 will be given in section 7.

From the viewpoint of dual Ginzburg-Landau theory, two constants m_1 and m_2 may be regarded as the coherence length m_ϕ and the penetration depth m_b (apart from a factor $\sqrt{2}$). Therefore, we can identify

$$m_1 \rightarrow m_\phi = 2\sqrt{\lambda}v, \quad m_2 \rightarrow m_b = \sqrt{2}g_mv, \quad (5.32)$$

which leads to the ratio given by

$$\frac{m_1^2}{m_2^2} = \frac{\lambda g^2}{8\pi^2} = 2\frac{\lambda}{g_m^2}. \quad (5.33)$$

In the Bogomolny limit, the coupling λ is solely given by the Yang-Mills coupling constant g as $\lambda = 8\pi^2/g^2$. The value of λ in the Bogomolny limit is smaller than that in the London limit, but it is still large $\lambda \sim 20$ even for $g \sim 2$, see section 7.

In the Bogomolny limit $m_1 = m_2$, the mass satisfies

$$m_1^2 = m_2^2 = m^2 = \gamma^2 = 2\eta^2. \quad (5.34)$$

The string tension in the Bogomolny limit is given by

$$\sigma_{st} \cong \frac{J^2 g^2}{2\pi} \rho^{-2} K \eta^2 = \frac{J^2 g^2}{4\pi} \rho^{-2} K m^2 @, \quad (5.35)$$

since $\ln(m_1^2/m_2^2) = \ln(1 + \sqrt{1 - 4\eta^4/\gamma^4}) - \ln(1 - \sqrt{1 - 4\eta^4/\gamma^4}) \cong 2\sqrt{1 - 4\eta^4/\gamma^4}$. In this limit, the monopole condensate reads

$$\langle k_\mu k_\mu \rangle = \frac{1}{4\pi^2} \left[2(\ln 4\pi - \gamma_E) + 1 - 2 \ln m^2 \right] m^6. \quad (5.36)$$

In the London limit, m_2 reduces to the mass m_b of the dual gauge field and m_1 to the diverging mass m_ϕ of the Higgs field,

$$m_2^2 \rightarrow \eta^2 (= m_b^2), \quad m_1^2 \rightarrow \frac{\gamma^4}{\eta^2} (= m_\phi^2 \rightarrow \infty). \quad (5.37)$$

Actually, in the near London case of type II, we have

$$\sigma_{st} \cong \frac{J^2 g^2}{2\pi} \rho^{-2} K \eta^2 \ln \frac{\gamma^2}{\eta^2}, \quad (5.38)$$

which diverges in the naive London limit $\gamma \rightarrow \infty$.

5.3 Type of dual superconductivity

In the LEET, the ratio η^4/γ^4 determines the type of the QCD vacuum as the dual superconductor,

$$\frac{\eta^4}{\gamma^4} = \frac{2\pi^2}{g^2 C_2 \sigma^2} \frac{f_2(\alpha)}{f_1(\alpha)^2} K. \quad (5.39)$$

If the ratio η^4/γ^4 is in the range $[0, 1/4)$, the QCD vacuum is the type II dual superconductor. Then $\sigma = 0$ seems to be excluded. If this ratio is zero, the QCD vacuum is the dual superconductor in the London limit. On the other hand, if the ratio η^4/γ^4 approaches $1/4$, the dual superconductor is on the border between type II and type I. The Bogomolny limit $m_1 = m_2$ is achieved if the ratio is $\eta^2/\gamma^2 = 1/2$. The function $f_2(\alpha)/f_1(\alpha)^2$ is positive for $\alpha > \alpha_0$ and it has a peak 0.21 at $\alpha \cong 2.65$, although it approaches zero slowly as α increases. The negative factor can come from $\ln \frac{M_A^2}{\mu^2}$ in K . We can reproduce both types of the dual superconductor by choosing μ . depending on the ratio M_A/μ .

Finally, we discuss how the above results change if we consider the effect of $\mathcal{D}[\partial]$.

$$\begin{aligned} D_m(p^2) \mathcal{D}[p]^2 &= \left(\frac{1}{p^2} - \frac{1}{\eta^2} + \frac{p^2}{\gamma^4} \right) \left(1 - \frac{p^2}{\eta^2} - \frac{p^4}{\gamma^4} \right)^2 \\ &= \frac{1}{p^2} - \frac{3}{\eta^2} + p^2 \left(\frac{3}{\eta^4} - \frac{1}{\gamma^4} \right), \end{aligned} \quad (5.40)$$

Therefore, the effect is equivalent to the following replacement of the parameters,

$$\eta^2 \rightarrow \frac{1}{3}\eta^2, \quad \frac{1}{\gamma^4} \rightarrow \frac{3}{\eta^4} - \frac{1}{\gamma^4}. \quad (5.41)$$

Consequently, m_1, m_2, χ are modified as

$$m_{1,2}^2 = \frac{\eta^2}{2 \left(1 - \frac{1}{3} \frac{\eta^4}{\gamma^4} \right)} \left(1 \pm \frac{2}{3} \sqrt{\frac{\eta^4}{\gamma^4} - \frac{3}{4}} \right), \quad (5.42)$$

$$\chi = \frac{1}{2} \frac{\eta^2}{\sqrt{\frac{\eta^4}{\gamma^4} - \frac{3}{4}}}. \quad (5.43)$$

For $m_{1,2}^2$ to be real and positive, η^4/γ^4 must be in the range $[3/4, 3)$. The type II theory lies in between the London limit $\eta/\gamma \rightarrow 0$ and the Bogomolny limit $\eta^4/\gamma^4 \rightarrow 3/4$. Therefore, the dual superconductivity of type II is excluded. This result suggests that the dual superconductivity of QCD is of type I or on the border between type I and type II. Therefore, the correction is important to determine the type of dual superconductivity. The above result will be compared with the lattice results in section 7.

A simple estimation is as follows. When $\alpha = 1, \sigma = 1$, η and γ are

$$\begin{aligned} \eta^2 &= 6 \frac{2\pi^2}{Ng^2} K M_A^2 = 10F(g) M_A^2, \\ \gamma^2 &= \frac{2\sqrt{30}}{\sqrt{Ng^2}} K^{1/2} M_A^2 = 10F(g)^{1/2} M_A^2, \end{aligned} \quad (5.44)$$

and the ratio is

$$\frac{\eta^4}{\gamma^4} = \frac{2\pi^2}{Ng^2} \frac{3}{5} K = \frac{2\pi^2}{Ng^2} \frac{3}{5} \left[1 + \frac{24}{11} \sigma^2 - \frac{Ng^2}{2\pi^2} \sigma^2 \left(\ln \frac{g^2}{16\pi^2} + 1 \right) \right] := F(g). \quad (5.45)$$

The function $F = F(g)$ is monotonically decreasing in g and goes into the negative region for $g > 9.6$ which is beyond the reach of the analyses of this paper. Finally, the string tension in SU(2) Yang-Mills theory is expressed as a function of the Yang-Mills coupling constant g as

$$\frac{\sigma_{st}}{M_A^2} = \frac{J^2 g^2}{2\pi} 5 \left(\frac{7}{6} \right)^2 K(g) \frac{F(g)}{\sqrt{F(g) - \frac{3}{4}}} \ln \frac{1 + \frac{2}{3} \sqrt{F(g) - \frac{3}{4}}}{1 - \frac{2}{3} \sqrt{F(g) - \frac{3}{4}}}, \quad (5.46)$$

where $F(g) := \frac{2\pi^2}{Ng^2} \frac{3}{5} K(g)$. More details will be given in a subsequent paper.

6 Confining string theory and string tension

Finally, we derive the confining string theory as a LEET of Gluodynamics. The final expression of the string action indicates the area law of the Wilson loop or the linear static interquark potential where the string tension is represented as a proportional constant. It is an old idea that the confining phase of gauge theories can be formulated as a string theory, see a review[44]. Especially, the large- N QCD might be exactly reformulated as a string theory[45]. For earlier approaches of QCD (gluon) string in the last century, see the references [46, 47, 48, 49, 50, 51, 52, 53, 25].

We can decompose the phase variable θ as $\theta := \theta^r + \theta^s$ where θ^r is the regular piece and θ^s the singular piece. Then it is shown [54] that the integration measure $\mathcal{D}\theta$ over the field θ factorizes into the product of measures, i.e., $\mathcal{D}\theta = \mathcal{D}\theta^r \mathcal{D}\theta^s$. The singularity of the phase of the scalar field ϕ just takes place at the string world sheet. The location of the world sheet, $x = x(\sigma)$, is determined by the condition $\phi(x(\sigma)) = 0$ which implies $|\phi(x)| = 0$ on the world sheet where the angle $\theta(x)$ is not determined uniquely. The singular piece $\theta^s(x)$ describes a configuration of the vortex string whose world-sheet obeys the equation,

$$\frac{1}{4\pi} \epsilon^{\mu\nu\rho\sigma} (\partial_\rho \partial_\sigma - \partial_\sigma \partial_\rho) \theta^s(x) = \Theta_{\mu\nu}(x). \quad (6.1)$$

On the other hand, the regular piece θ^r describes a single-valued fluctuation around the string configuration just mentioned above. From the correspondence (6.1), the integration over θ^s corresponds to the integration over the world-sheet $x_\mu(\sigma)$ of the string, so that the integration measure is transformed as

$$\mathcal{D}\theta^s(x) \rightarrow \mathcal{D}x_\mu(\sigma) J[x], \quad (6.2)$$

where $J[x]$ is the Jacobian for the change of the integration variables.²¹

²¹The Jacobian $J[x]$ has been evaluated in [54] for the world-sheet of the closed string. The Jacobian exactly cancels the conformal anomaly. Such a possibility was already suggested in the paper [60]. Therefore, the vortex string might be self-consistent in $D = 4$ space-time dimensions at least in the long-distance limit. See also reference [56, 57, 58, 55].

We return to the expression of the VEV of the Wilson loop (2.118) for the action (2.117), i.e.,

$$S_{APEGT} = (h, \Delta D_m h) + 2Jg\rho^{-1}K^{1/2}(h, \tilde{\Theta}). \quad (6.3)$$

Performing the Gaussian integration over the Kalb-Ramond field $h_{\mu\nu}$, we obtain the an action written in terms of the vorticity tensor,,

$$S_{cs} = \int d^4x \int d^4y (2Jg\rho^{-1}K^{1/2})^2 \tilde{\Theta}_{\mu\nu}(x) \left(\frac{\chi}{\Delta - m_2^2} - \frac{\chi}{\Delta - m_1^2} \right) (x, y) \tilde{\Theta}^{\mu\nu}(y), \quad (6.4)$$

for the expectation value of the Wilson loop,

$$\langle W(C) \rangle_{YM} = Z_{cs}[C]/Z_{cs}[0], \quad Z_{cs}[C] = \int \mathcal{D}x_\mu(\sigma) J[x] \exp\{iS_{cs}[x]\}. \quad (6.5)$$

Here m_1, m_2, χ are the same as those defined in section 5,

$$m_{1,2}^2 := \frac{\gamma^4}{2\eta^2} \left(1 \pm \sqrt{1 - 4\eta^4/\gamma^4} \right) \quad (m_1 \geq m_2), \quad (6.6)$$

$$\chi := \frac{m_1^2 m_2^2}{m_1^2 - m_2^2} = \frac{\eta^2}{\sqrt{1 - 4\eta^4/\gamma^4}}. \quad (6.7)$$

Let $\sigma = (\sigma^1, \sigma^2)$ be a two-dimensional coordinate on the world sheet $x_\mu = x_\mu(\sigma)$. Then the infinitesimal surface element,

$$dS_{\mu\nu}(x(\sigma)) = \sqrt{g(\sigma)} t_{\mu\nu}(\sigma) d^2\sigma, \quad (6.8)$$

is expressed by the determinant $g(\sigma) = \det ||g_{ab}(\sigma)||$ calculated from the induced metric tensor of the surface defined by

$$g_{ab}(\sigma) = \partial_a x_\mu(\sigma) \partial_b x_\mu(\sigma) \quad (6.9)$$

with the derivative,

$$\partial_a = \frac{\partial}{\partial \sigma^a} \quad (a = 1, 2), \quad (6.10)$$

and the so-called extrinsic curvature tensor of the surface,

$$t_{\mu\nu}(\sigma) = \frac{\epsilon^{ab}}{\sqrt{g(\sigma)}} \partial_a x_\mu(\sigma) \partial_b x_\nu(\sigma). \quad (6.11)$$

Given the theory with the action of the form,

$$S = (\Theta, \frac{\kappa}{\Delta^2 - m^2} \Theta), \quad (6.12)$$

the low-energy limit is obtained by performing a derivative expansion of this action. The derivative expansion is equivalent to an expansion in powers of $1/m$. This procedure is well known in the literatures, see e.g. [61, 63, 64, 65, 66, 67, 68] and Appendix G. The result is the Nambu-Goto action with a rigidity term,

$$S_{cs} = \sigma_{st}^0 \int_S d^2\sigma \sqrt{g} + \frac{1}{\alpha_0} \int d^2\sigma \sqrt{g} g^{ab} \partial_a t_{\mu\nu} \partial_b t^{\mu\nu} + \kappa_t \int d^2\sigma \sqrt{g} R + \dots \quad (6.13)$$

The string tension of the Nambu-Goto term is given by

$$\sigma_{st}^0 = \frac{\kappa}{4\pi} K_0\left(\frac{m}{\Lambda}\right), \quad (6.14)$$

where $K_0(x)$ is the modified Bessel function and Λ is the ultraviolet cut-off.²² Moreover, it has been shown that the coefficient of the extrinsic curvature term is a negative constant,

$$\frac{1}{\alpha_0} = -\frac{1}{128\pi} < 0, \quad (6.15)$$

which is independent of κ, m, Λ .

In the naive confining string theory based on the action (6.13), the string tension diverges $\sigma_{st} \rightarrow \infty$ as $\Lambda \rightarrow \infty$, since $K_0(0) = +\infty$. This pathology can be automatically avoided in the confining string theory derived in this paper. It is easy to see that the rigidity term cancels in the confining string theory (6.4). Therefore, the action (6.4) is cast into the confining string action,

$$S_{cs} = \sigma_{st} \int_S d^2\sigma \sqrt{g} + \frac{1}{\alpha_0} \int d^2\sigma \sqrt{g} g^{ab} \partial_a t_{\mu\nu} \partial_b t^{\mu\nu} + \kappa_t \int d^2\sigma \sqrt{g} R + \dots, \quad (6.16)$$

with the string tension,

$$\sigma_{st} = (2Jg\rho^{-1}K^{1/2})^2 \frac{\chi}{4\pi} \left[K_0\left(\frac{m_2}{\Lambda}\right) - K_0\left(\frac{m_1}{\Lambda}\right) \right]. \quad (6.17)$$

Note that the asymptotic behavior of the modified Bessel function $K_0(z)$ for $z \ll 1$ is given by

$$K_0(z) \cong -(\gamma_E + \ln \frac{z}{2}) = \ln \frac{2e^{-\gamma_E}}{z}, \quad (6.18)$$

with γ_E being Euler's constant $\gamma_E = 0.5772\dots$. Thus, for sufficiently large Λ , we obtain the Λ -independent *finite* result,²³

$$\sigma_{st} \cong \frac{(2Jg)^2}{4\pi} \rho^{-2} K \chi \ln\left(\frac{m_1}{m_2}\right), \quad (6.19)$$

²²In the setting up of this paper, the role of the ultraviolet cutoff Λ is played by the Higgs mass $m_H/\sqrt{2} = m_1$, as shown below.

²³The string tension is a free energy per unit length of the string. It is well known that the free energy has a logarithmic dependence in the Ginzburg-Landau theory. The London limit corresponds to $m_1 \rightarrow \infty$. Hence, the string tension reduces to the expression, $\sigma_{st} = (2Jg\rho^{-1}K^{1/2})^2 \frac{\chi}{4\pi} K_0\left(\frac{m_2}{\Lambda}\right)$, since $K_0(m_1/\Lambda) \rightarrow 0$.

where $J = 1/2$ for the fundamental and $J = 1$ for the adjoint quark (sources) in the case of $SU(2)$, and $J = 1/3$ for the fundamental quark in the case of $SU(3)$. The string tension just obtained agrees with that obtained from the monopole action (5.18). The mass m_1 can be viewed as a dual Higgs mass m_H and $1/m_1$ corresponds to a finite thickness of the string. The limit $m_1 \rightarrow \infty$ corresponds to the London limit, the thin string. This is consistent with a fact that the naive confining string theory with the string tension (6.14) is obtained from the DAH model in the London limit.

When the theory is near the London limit $m_1 \gg m_2$, the expression of the string tension reduces to

$$\sigma_{st} \cong \frac{(2Jg)^2}{8\pi} \rho^{-2} K m_b^2 \ln \left(\frac{m_H}{m_b} \right)^2, \quad (6.20)$$

which is similar to the result obtained by Suganuma, Sasaki and Toki [43]. In this case, it is again cast into the form,

$$\sigma_{st} \cong \frac{(2Jg)^2}{4\pi} \rho^{-2} K m_b^2 K_0 \left(\sqrt{2} \frac{m_b}{m_H} \right), \quad (6.21)$$

which agrees with the result of Suzuki [8] up to a numerical factor. Thus our results agree with those predicted based on the hypothetical DGL theory. This fact also supports that the DGL theory is one of the LEET's of Gluodynamics.

The coefficient of the rigidity term is a negative constant,

$$\alpha_0^{-1} = -J^2 g^2 \rho^{-2} K \frac{1}{\pi} < 0. \quad (6.22)$$

Finally, the κ is a positive constant,

$$\kappa = \frac{2}{3} J^2 g^2 \rho^{-2} K \frac{1}{\pi} = -\frac{2}{3} \alpha_0^{-1} > 0. \quad (6.23)$$

String theory was originally developed as dual resonance models to explain hadronic physics. It was almost abandoned after the invention of QCD and the discovery of asymptotic freedom. Nevertheless, string theory might be useful by offering an alternative but tractable method of solving the strong coupling problem at long distance or low energies to which QCD has not yet given the definite answer. Indeed, it is known that ordinary strings have the notorious troubles such as tachyons and conformal anomaly (critical dimensions). However, they disappear asymptotically at large distance as shown e.g. by Olesen[58]. In other words, a string theory becomes a consistent model at large distances, although it is in a strict sense inconsistent in four dimensions. Therefore, the large-distance QCD can be described by a string model. Of course, such a description breaks down at some distance, since QCD does not contain tachyons and is Lorentz invariant in spacetime dimensions D less than or equal to four. In view of this, it should be worthwhile to mention that the static potential for the Nambu-Goto model (i.e., bosonic string model with Nambu-Goto action) has been computed to leading order in the $1/D$ expansion by Alvarez [56] and exactly in the whole range of R by Arvis [57] in arbitrary dimension D ,

$$V(R) = \sigma_{st} (R^2 - R_c^2)^{1/2}, \quad R_c = \sqrt{\frac{\pi(D-2)}{12\sigma_{st}}}. \quad (6.24)$$

In the long-distance region $R > R_c$, $V(R)$ is expanded into

$$V(R) = \sigma_{st}R - \frac{\pi(D-2)}{24R} + (1/R^3). \quad (6.25)$$

For the large R , therefore, the static interquark potential is given by the linear potential where the string tension σ_{st} is the proportional coefficient. Moreover, the static potential has an additional long-distance Coulomb term which agrees with the earlier observation of Lüscher, Symanzik and Weisz [59]. The long-distance Coulomb term is the universal term depending only on the dimensionality of spacetime due to the transverse displacement x_T of the string with fixed end points, under the assumption that x_T is the only relevant dynamical variable at large distances (This is not the case for the superstring [58]). The second term should not be confused with the short-distance Coulomb potential which is consistent with the asymptotic freedom of QCD. We see that the expression of $V(R)$ loses the meaning at short distance $R < R_c$.

In this paper we have shown that the Nambu-Goto action can be a piece of the effective string action as a LEET of QCD. Therefore, QCD should possess a long-distance Coulomb potential. If the string theory which is capable of describing the whole energy range of QCD exists, the string theory must reduce in the long-distance limit to the above string theory derived in this paper.

The large N expansion can give another link between QCD and string theory, as pointed out by 't Hooft[45]. The leading order of this expansion is some kind of free string theory that has yet to be identified. In a free string theory, the surfaces in the sum should be dominated by smooth surfaces with no surface tension and without self-intersections. The Nambu-Goto model describes fundamental string without a transverse extension. The string describing color electric flux tubes in QCD must be thick strings with a fundamental transverse length scale m_H^{-1} (The London limit $m_H \rightarrow \infty$ corresponds to a thin string) [55]. If so, the string action should be responsible to the bending rigidity due to the finite width of the string. In order to take into account these features, the string model with the extrinsic curvature (the so-called rigid string) has been introduced by Polyakov [47] and Kleinert [48]. The extrinsic curvature stiffness was expected to suppress the crumpled surface with a large number of self-intersections. However, it turns out that the new term is infrared irrelevant, see [64] and references therein for more details.

Recently, new string theories (so-called the confining string theory) of describing the confining phase in gauge theories were proposed by Polyakov [25] and by Kleinert and Chervyakov [61]. The confining string theory has a non-local interaction between world-sheet elements and a negative stiffness. The confining string theories are very promising, since they seem to solve all the problems of rigid strings. Especially, a negative stiffness is crucial in order to match the correct high-temperature behavior of large N QCD [62, 61]. The confining string theory can be explicitly derived for Abelian gauge theories, compact $U(1)$ gauge theory [63], Abelian Higgs model[66] and so on, see a review [68]. In this paper we have derived the confining string with a negative stiffness directly from QCD at least in the low-energy regime. This is performed by integrating out the antisymmetric tensor field in the improved version of the APEGT (originally proposed by the author in the paper[11]) which was derived

directly from QCD. The action realizes explicitly the necessary zig-zag invariance of confining string [69, 70].

7 Parameter fitting for numerical estimation

For the numerical estimation of physical quantities, we can use the following values which seem to be mutually consistent.

7.1 Dual Ginzburg-Landau theory

We use the values suggested in [8] For the dimensionful quantities,

$$m_b = \sqrt{2}g_mv = \sqrt{2}\frac{4\pi}{g}v \sim 0.88\text{GeV}, \quad m_H = 2\sqrt{\lambda}v \sim 18\text{GeV}, \quad v \sim 0.1\text{GeV}, \quad (7.1)$$

and for the dimensionless coupling constants,

$$\alpha_s := \frac{g^2}{4\pi} \sim 0.24 \quad (g \sim 1.7). \quad \lambda \sim 8 \times 10^3. \quad (7.2)$$

These values are consistent with the string tension,

$$\sigma_{st} \sim (0.42\text{GeV})^2 \sim 0.18(\text{GeV})^2. \quad (7.3)$$

The off-diagonal gluon mass obtained by Monte Carlo simulation [23] for $SU(2)$ is

$$M_A = 1.2\text{GeV}. \quad (7.4)$$

7.2 Monopole action

In the paper [24] the following lattice monopole action was adopted,

$$S[k] = \sum_{s,s',\mu} k_\mu(s) D_0(s-s') k_\mu(s'), \quad (7.5)$$

where D_0 is parameterized by three parameters as

$$D_0(s-s') = \bar{\alpha}\delta_{s,s'} + \bar{\beta}\Delta_L^{-1}(s-s') + \bar{\gamma}\Delta_L(s-s'), \quad (7.6)$$

with the lattice Laplacian $\Delta_L(s-s') := -\partial\partial'$. This leads to $D_0(p) = \bar{\alpha} + \bar{\beta}/p^2 + \bar{\gamma}p^2$. Its inverse is

$$D_0^{-1}(p) = \kappa \left(\frac{m_1^2}{p^2 + m_1^2} - \frac{m_2^2}{p^2 + m_2^2} \right), \quad (7.7)$$

where

$$m_1^2 + m_2^2 = \bar{\alpha}/\bar{\gamma}, \quad m_1^2 m_2^2 = \bar{\beta}/\bar{\gamma}, \quad \kappa := \bar{\gamma}^{-1}/(m_1^2 - m_2^2). \quad (7.8)$$

The string tension is obtained as $\sigma_{tot} = \sigma_{cl} + \sigma_q$ with

$$\sigma_{cl} = \frac{\pi}{2} \kappa \ln \frac{m_1}{m_2}, \quad (7.9)$$

and σ_q being negligible. The results of [24] are

$$m_1 \cong 1.0 \times 10^4, \quad m_2 \cong 12, \quad \kappa = 4.83, \quad (7.10)$$

and the parameters of the monopole action are obtained as

$$\bar{\alpha} = 0.207(0.435), \quad \bar{\beta} = 2.49, \quad \bar{\gamma} = 2.07(9.15) \times 10^{-5}, \quad (7.11)$$

and

$$\sigma_{phys} \cong (0.44\text{GeV})^2, \quad \sigma_{phys}/\sigma_{cl} \sim (1.4)^{-2} \sim 0.51. \quad (7.12)$$

Note that our parameterization is

$$m_1^2 + m_2^2 = \gamma^4/\eta^2, \quad m_1^2 m_2^2 = \gamma^4, \quad m_1^2 - m_2^2 = (\gamma^4/\eta^2)\sqrt{1 - 4\eta^4/\gamma^4}. \quad (7.13)$$

Hence the correspondence of the parameters in our theory to the lattice result [24] are given by

$$\eta^2 \rightarrow \bar{\beta}/\bar{\alpha} = 10, \quad \gamma^4 \rightarrow \bar{\beta}/\bar{\gamma} = 10^5. \quad (7.14)$$

Hence we obtain

$$m_1 \cong 10^2, \quad m_2 \cong 1, \quad \chi = 10. \quad (7.15)$$

7.3 Confining string

The confining string action has the following parameters (see section 2.1 of the paper [68]),

$$\sigma \cong 0.2(\text{GeV})^2, \quad \kappa_t \cong 0.003, \quad \frac{1}{\alpha_0} \cong -0.005. \quad (7.16)$$

They are obtained from the so-called correlation length of the vacuum and the gluon condensate,

$$T_g \cong 0.65(\text{GeV})^{-1} (T_g^{-1} \cong 1.5\text{GeV}), \quad \alpha_s \langle (\mathcal{F}_{\mu\nu}^A)^2 \rangle \sim 0.038(\text{GeV})^2. \quad (7.17)$$

8 Conclusion and discussion

In this paper, by improving the strategy suggested in the previous paper [11], we have derived three equivalent LEET's of Gluodynamics, i.e., dual Ginzburg-Landau theory, magnetic monopole theory and confining string theory. Although each of them has already been proposed and analyzed as a LEET by various authors, we have given a first-principle derivation of these theories directly from Gluodynamics, i.e., Yang-Mills theory. In other words, we have shown their equivalence in this paper as is obvious from the derivation. Especially, we have shown that the monopole condensation occurs due to non-zero mass of off-diagonal gluons.

The very origin of our nontrivial results is reduced to quantum correction to the diagonal fields a_μ and $B_{\mu\nu}$ arising from the massive off-diagonal gluons and off-diagonal ghosts. This is a novel viewpoint examined in this paper in analyzing the low-energy Gluodynamics. In the conventional analytical approaches, the off-diagonal

components are completely neglected from the beginning by virtue of the infrared Abelian dominance. However, we notice that it is the off-diagonal components that convey the characteristic properties of the original non-Abelian gauge theory to the LEET written in terms of the diagonal field alone. The reproduction of the β function of the original Yang-Mills theory in the LEET is a good example of this fact.

As a mechanism for mass generation of the off-diagonal component, we have used the ghost–anti-ghost condensation caused by the quartic ghost interaction in the modified MA gauge (section 2.5). To author’s knowledge, any other analytical derivation of off-diagonal gluon mass is not known for the Yang-Mills theory in the MA gauge. However, the following steps after section 2.5 can be performed irrespective of the origin of the off-diagonal gluon mass, once we regard the massive off-diagonal gluons. Therefore, even if the off-diagonal gluons become massive due to other mechanism, we obtain the same LEET’s as given in this paper.

The LEET’s of the $SU(N)$ Yang-Mills theory have been obtained by way of the APEGT. The APEGT is bilinear in the diagonal fields, a_μ and $B_{\mu\nu}$. This result is regarded as the leading order result of the large N expansion. The (higher-order) correction terms are suppressed by a factor N^{-2} . Since we have obtained the confining string theory by rewriting the APEGT, this is consistent with the claim that the Yang-Mills theory in the large N limit is equivalent to a certain string theory [45]. Therefore, our result may shed more light on the relationship between the gauge theory and the string theory. However, we have used the derivative (or low-energy) expansion or weak field approximation [25] to derive the confining string theory. Therefore, it is not clear where the multi-valuedness of the confining string action comes from. Multi-valuedness is considered as a reflection of the compactness of the residual Abelian gauge group which is embedded in the original compact non-Abelian gauge group. This problem will be discussed elsewhere.

By way of LEET’s, we have succeeded to calculate the string tension of QCD (gluon) string. The non-zero value of the string tension implies area law of the Wilson loop, i.e., quark confinement. However, the string tension obtained in this way depends on the parameters ρ, σ which were introduced to rewrite the Yang-Mills theory into the dual Abelian gauge theory. It is possible for the string tension to be independent of the renormalization scale μ . In a subsequent paper [72], we will discuss in detail the issue how those parameters are determined within the same framework as that proposed in this paper. We will give a simple model (toy model for the gauge theory) in which the role of parameters introduced in the auxiliary field formalism is clarified in more tractable way. We will also give more quantitative argument so that the LEET’s derived in this paper can reproduce the experimental values of physical quantities and predict unobserved quantities, e.g., the glueball mass. We hope to discuss the spontaneous chiral symmetry breaking which is expected to occur simultaneously with quark confinement.

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A Useful formulae

A.1 Structure constants

Note that

$$f^{ACD} f^{BCD} = C_2 \delta^{AB}, \quad (\text{A.1})$$

where C_2 is called the quadratic Casimir (operator). This implies that

$$f^{icd} f^{jcd} = C_2 \delta^{ij}, \quad (\text{A.2})$$

since the structure constant with two and three diagonal indices are zero, $f^{ijc} = 0 = f^{ijk}$ according to $f^{ABC} = -2\sqrt{-1}\text{Tr}\{[T^A, T^B]T^C\}$. Combining the identity,

$$f^{iac} f^{ibc} = \delta^{ab}, \quad (\text{A.3})$$

with (A.1) leads to

$$f^{acd} f^{bcd} = (C_2 - 2)\delta^{ab}. \quad (\text{A.4})$$

A.2 Differential forms

For the p-form,

$$\omega := \frac{1}{p!} \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \quad (\text{A.5})$$

the dual form $*\omega$ in four dimensional Minkowski space is defined by

$$*\omega := \frac{1}{(4-p)!} * \omega_{\mu_1 \dots \mu_{4-p}} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{4-p}}, \quad (\text{A.6})$$

$$\omega_{\mu_1 \dots \mu_{4-p}} := \frac{1}{p!} \epsilon_{\mu_1 \dots \mu_{4-p} \nu_1 \dots \nu_p} \omega^{\nu_1 \dots \nu_p}. \quad (\text{A.7})$$

The identity,

$$\epsilon_{\mu_1 \dots \mu_p \alpha_1 \dots \alpha_{4-p}} \epsilon^{\mu_1 \dots \mu_p \beta_1 \dots \beta_{4-p}} = -g^{-1} p! (4-p)! \delta_{[\beta_1}^{\alpha_1} \dots \delta_{\beta_{4-p}] }^{\alpha_{4-p}}, \quad (\text{A.8})$$

leads to

$$* * \omega = g^{-1} (-1)^p \omega, \quad (\text{A.9})$$

since

$$\epsilon^{\mu_1 \dots \mu_p \beta_1 \dots \beta_{4-p}} = g^{-1} \epsilon_{\mu_1 \dots \mu_p \beta_1 \dots \beta_{4-p}}, \quad g := \det(g_{\mu\nu}). \quad (\text{A.10})$$

Note that $g = -1$ for Minkowski spacetime with a Lorentz metric, while $g = 1$ for Euclidean space.

A.3 Integration formula by dimensional regularization

Define

$$\epsilon := 2 - \frac{D}{2} \quad (\text{A.11})$$

In Minkowski space,

$$\int \frac{d^D k}{i(2\pi)^D} \ln(m^2 + 2p \cdot k - k^2) = -\frac{\Gamma(\epsilon - 2)}{(4\pi)^{2-\epsilon}} (m^2 + p^2)^{2-\epsilon}, \quad (\text{A.12})$$

$$\int \frac{d^D k}{i(2\pi)^D} \frac{1}{(m^2 + 2p \cdot k - k^2)^a} = \frac{\Gamma(\epsilon + a - 2)}{(4\pi)^{2-\epsilon} \Gamma(a)} (m^2 + p^2)^{2-a-\epsilon}, \quad (\text{A.13})$$

$$\int \frac{d^D k}{i(2\pi)^D} \frac{k_\mu}{(m^2 + 2p \cdot k - k^2)^a} = \frac{\Gamma(\epsilon + a - 2)}{(4\pi)^{2-\epsilon} \Gamma(a)} p_\mu (m^2 + p^2)^{2-a-\epsilon}, \quad (\text{A.14})$$

$$\begin{aligned} & \int \frac{d^D k}{i(2\pi)^D} \frac{k_\mu k_\nu}{(m^2 + 2p \cdot k - k^2)^a} \\ &= \frac{1}{(4\pi)^{2-\epsilon} \Gamma(a)} \left[\Gamma(\epsilon + a - 2) p_\mu p_\nu (m^2 + p^2)^{2-a-\epsilon} - \Gamma(\epsilon + a - 3) \frac{1}{2} g_{\mu\nu} (m^2 + p^2)^{3-a-\epsilon} \right], \end{aligned} \quad (\text{A.15})$$

where m^2 is understood as $m^2 - i\delta$, $\delta > 0$, i.e., $\text{Im}(m^2) < 0$.

A.4 Gamma function

The Laurent expansion of the Gamma function is as follows.

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \quad (\text{A.16})$$

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma_E - 1 + O(\epsilon), \quad (\text{A.17})$$

$$\Gamma(\epsilon - 2) = \frac{1}{\epsilon} - \gamma_E + \frac{3}{2} + O(\epsilon), \quad (\text{A.18})$$

$$\Gamma(\epsilon - n) = \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} - \gamma_E + \sum_{k=1}^n \frac{1}{k} \right] + O(\epsilon), \quad (\text{A.19})$$

where γ_E is Euler's constant $\gamma_E = 0.5772 \dots$.

B Derivation of a version of non-Abelian Stokes theorem

In an expression of the non-Abelian Stokes theorem,

$$W(C) = \int d\mu_C(\xi) \exp \left[ig \int_{S_C} dS^{\mu\nu} f_{\mu\nu}^\xi(x) \right], \quad (\text{B.1})$$

the argument of the exponential is rewritten as follows. The target space coordinate of the surface spanned by the Wilson loop C is denoted by $x(\sigma)$ where σ is the world sheet coordinate. The surface integral is rewritten in terms of the vorticity tensor as

$$\begin{aligned}
\frac{1}{2} \int_{S_C} dS^{\mu\nu}(x(\sigma)) f_{\mu\nu}(x(\sigma)) &= \int d^4x \Theta^{\mu\nu}(x) f_{\mu\nu}(x) & (B.2) \\
&:= (\Theta, f) = (*\Theta, *f) \\
&= (*\Theta, \Delta^{-1}(d\delta + \delta d) * f) \\
&= (*\Theta, \Delta^{-1}d\delta * f) + (*\Theta, \Delta^{-1}\delta d * f) \\
&= (\delta\Delta^{-1} * \Theta, \delta * f) + (\Theta, *\Delta^{-1}\delta * \delta f) \\
&= (\delta\Delta^{-1} * \Theta\delta * f) + (\Theta, \Delta^{-1}d\delta f) \\
&= (\delta\Delta^{-1} * \Theta, k) + (\Delta^{-1}\delta\Theta, j), & (B.3)
\end{aligned}$$

where $k := \delta * f$ is the magnetic monopole current and $j := \delta f$ is the electric current. Assuming the absence of the electric source $j = 0$, we obtain

$$W(C) = \int d\mu_C(\xi) \exp \left[ig(\Xi, k^\xi) + ig(N, j^\xi) \right], \quad (B.4)$$

where N is one-form defined by

$$\Xi := *d\Theta\Delta^{-1} = \delta * \Theta\Delta^{-1}, \quad N := \delta\Theta\Delta^{-1}. \quad (B.5)$$

with the components,

$$\Xi^\mu(x) = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu^x \int d^4y \Theta_{\rho\sigma}(y) \Delta^{-1}(y-x) \quad (B.6)$$

$$= \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \partial_\nu^x \int_S d^2S_{\rho\sigma}(x(\sigma)) \Delta^{-1}(x(\sigma)-x), \quad (B.7)$$

$$N^\mu(x) = \partial_\nu^x \int d^4y \Theta^{\mu\nu}(y) \Delta^{-1}(y-x) \quad (B.8)$$

$$= \frac{1}{2} \partial_\nu^x \int_S d^2S^{\mu\nu}(x(\sigma)) \Delta^{-1}(x(\sigma)-x). \quad (B.9)$$

C Calculation of the vacuum polarization for tensor fields

We shall evaluate the vacuum polarization of the tensor field B . The contribution from Fig. 2 is calculated from

$$\begin{aligned}
\Pi_{\mu\nu, \alpha\beta}^{ij}(k) &:= \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} D_{\sigma_1\sigma_2}(p) \delta^{d_1d_2} [-2\sigma g f^{ic_1d_1} I_{\mu\nu, \rho_1\sigma_1}] \\
&\quad \times D_{\rho_1\rho_2}(p+k) \delta^{c_1c_2} [-2\sigma g f^{jc_2d_2} I_{\alpha\beta, \rho_2\sigma_2}], & (C.1)
\end{aligned}$$

where

$$I_{\mu\nu, \alpha\beta} := \frac{1}{2} (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}), \quad (C.2)$$

and the off-diagonal massive gluon propagator $D_{\mu\nu}^{ab}(k) = \delta^{ab}D_{\mu\nu}(k)$ is given by

$$D_{\mu\nu}(k) := \frac{1}{k^2 - M_A^2} \left[g_{\mu\nu} - (1 - \alpha) \frac{k_\mu k_\nu}{k^2 - \alpha M_A^2} \right] \quad (\text{C.3})$$

$$= \frac{1}{k^2 - M^2} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2} \right) + \frac{k_\mu k_\nu}{M^2} \frac{1}{k^2 - \alpha M^2} \quad (\text{C.4})$$

$$= \frac{1}{M^2} \left[\frac{M^2 g_{\mu\nu} - k_\mu k_\nu}{k^2 - M^2} + \frac{k_\mu k_\nu}{k^2 - \alpha M^2} \right]. \quad (\text{C.5})$$

Hence the additional term to the APEGT is given by

$$\delta_{(1)}^c \mathcal{L}_{APEGT} = \int \frac{d^4 k}{(2\pi)^4} * B_{\mu\nu}^i(k) \Pi_{\mu\nu, \alpha\beta}^{ij}(k) * B_{\alpha\beta}^j(-k). \quad (\text{C.6})$$

Using the identity,

$$\delta^{d_1 d_2} f^{ic_1 d_1} \delta^{c_1 c_2} f^{jc_2 d_2} = f^{ic_1 d_1} f^{jc_1 d_1} = C_2 \delta^{ij}, \quad (\text{C.7})$$

we obtain

$$\Pi_{\mu\nu, \alpha\beta}^{ij}(k) := \frac{[-2\sigma g]^2}{2} C_2 \delta^{ij} I_{\mu\nu, \rho_1 \sigma_1} I_{\alpha\beta, \rho_2 \sigma_2} \int \frac{d^4 p}{(2\pi)^4} D_{\sigma_1 \sigma_2}(p) D_{\rho_1 \rho_2}(p+k) \quad (\text{C.8})$$

$$= \frac{2\sigma^2 g^2 C_2 \delta^{ij}}{M^4} I_{\mu\nu, \rho_1 \sigma_1} I_{\alpha\beta, \rho_2 \sigma_2} \int \frac{d^4 p}{(2\pi)^4} \left[\frac{M^2 g_{\sigma_1 \sigma_2} - p_{\sigma_1} p_{\sigma_2}}{p^2 - M^2} \frac{M^2 g_{\rho_1 \rho_2} - (p+k)_{\rho_1} (p+k)_{\rho_2}}{(p+k)^2 - M^2} \right. \quad (\text{C.9})$$

$$+ \frac{M^2 g_{\sigma_1 \sigma_2} - p_{\sigma_1} p_{\sigma_2}}{p^2 - M^2} \frac{(p+k)_{\rho_1} (p+k)_{\rho_2}}{(p+k)^2 - \alpha M^2} \quad (\text{C.10})$$

$$+ \frac{p_{\sigma_1} p_{\sigma_2}}{p^2 - \alpha M^2} \frac{M^2 g_{\rho_1 \rho_2} - (p+k)_{\rho_1} (p+k)_{\rho_2}}{(p+k)^2 - M^2} \quad (\text{C.11})$$

$$\left. + \frac{p_{\sigma_1} p_{\sigma_2}}{p^2 - \alpha M^2} \frac{(p+k)_{\rho_1} (p+k)_{\rho_2}}{(p+k)^2 - \alpha M^2} \right]. \quad (\text{C.12})$$

Using the Feynman parameter formulas, two denominators are combined into one denominator, e.g.,

$$\frac{1}{p^2 - \alpha M^2} \frac{1}{(p+k)^2 - \beta M^2} = \int_0^1 dx \frac{1}{[p^2 + 2xk \cdot p + \{xk^2 - (x\beta - x\alpha + \alpha)M^2\}]^2}, \quad (\text{C.13})$$

we have

$$\begin{aligned} \Pi_{\mu\nu, \alpha\beta}^{ij}(k) &= \frac{2\sigma^2 g^2 C_2 \delta^{ij}}{M^4} I_{\mu\nu, \rho_1 \sigma_1} I_{\alpha\beta, \rho_2 \sigma_2} \int_0^1 dx \left[\int \frac{d^4 p}{(2\pi)^4} \frac{[M^2 g_{\sigma_1 \sigma_2} - p_{\sigma_1} p_{\sigma_2}][M^2 g_{\rho_1 \rho_2} - (p+k)_{\rho_1} (p+k)_{\rho_2}]}{[p^2 + 2xk \cdot p + \{xk^2 - M^2\}]^2} \right] \quad (\text{C.14}) \end{aligned}$$

$$+ \int \frac{d^4 p}{(2\pi)^4} \frac{[M^2 g_{\sigma_1 \sigma_2} - p_{\sigma_1} p_{\sigma_2}][(p+k)_{\rho_1}(p+k)_{\rho_2}]}{[p^2 + 2xk \cdot p + \{xk^2 - (x\alpha - x + 1)M^2\}]^2} \quad (C.15)$$

$$+ \int \frac{d^4 p}{(2\pi)^4} \frac{[p_{\sigma_1} p_{\sigma_2}][M^2 g_{\rho_1 \rho_2} - (p+k)_{\rho_1}(p+k)_{\rho_2}]}{[p^2 + 2xk \cdot p + \{xk^2 - (x - x\alpha + \alpha)M^2\}]^2} \quad (C.16)$$

$$+ \int \frac{d^4 p}{(2\pi)^4} \frac{[p_{\sigma_1} p_{\sigma_2}][(p+k)_{\rho_1}(p+k)_{\rho_2}]}{[p^2 + 2xk \cdot p + \{xk^2 - \alpha M^2\}]^2} \Big]. \quad (C.17)$$

The momentum integration can be performed by making use of the formula in Appendix A where $\epsilon := 2 - D/2$. After straightforward but tedious calculations, we are lead to

$$\begin{aligned} & \Pi_{\mu\nu,\alpha\beta}^{ij}(k) \\ = & -2C_2\sigma^2 g^2 \delta^{ij} i I_{\mu\nu,\alpha\beta} \int_0^1 dx \frac{(4\pi)^\epsilon}{(4\pi)^2 \Gamma(2)} \\ & \times \left\{ \Gamma(\epsilon)[M^2 - x(1-x)k^2]^{-\epsilon} + M^{-2}\Gamma(-1+\epsilon)[M^2 - x(1-x)k^2]^{1-\epsilon} \right. \\ & - \frac{1}{2}M^{-2}\Gamma(-1+\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{1-\epsilon} \\ & \left. - \frac{1}{2}M^{-2}\Gamma(-1+\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{1-\epsilon} \right\} \\ & - 2C_2\sigma^2 g^2 \delta^{ij} i \frac{1}{2}k^2 (I - P)_{\mu\nu,\alpha\beta} \int_0^1 dx \frac{(4\pi)^\epsilon}{(4\pi)^2 \Gamma(2)} \\ & \times \left\{ -M^{-2}\Gamma(\epsilon)[M^2 - x(1-x)k^2]^{-\epsilon}(2x^2 + 2x + 1) \right. \\ & \left. - \frac{1}{2}M^{-4}\Gamma(-1+\epsilon)[M^2 - x(1-x)k^2]^{1-\epsilon} \right. \end{aligned} \quad (C.18)$$

$$+ M^{-2}\Gamma(\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{-\epsilon}(x^2 + 2x + 1) \\ + \frac{1}{2}M^{-4}\Gamma(-1+\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{1-\epsilon} \quad (C.19)$$

$$+ M^{-2}\Gamma(\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{-\epsilon}x^2 \\ + \frac{1}{2}M^{-4}\Gamma(-1+\epsilon)[\{\alpha + (1-\alpha)x\}M^2 - x(1-x)k^2]^{1-\epsilon} \quad (C.20)$$

$$\left. - \frac{1}{2}M^{-4}\Gamma(-1+\epsilon)[\alpha M^2 - x(1-x)k^2]^{1-\epsilon} \right\}, \quad (C.21)$$

where the fourth term does not give the term proportional to $I_{\mu\nu,\alpha\beta}$. Here we have used the following properties of the projection operator,

$$I_{\mu\nu,\alpha\beta} = -I_{\nu\mu,\alpha\beta} = I_{\alpha\beta,\mu\nu} = -I_{\mu\nu,\beta\alpha}, \quad (C.22)$$

$$I_{\mu\nu,\rho\sigma} I_{\rho\sigma,\alpha\beta} = I_{\mu\nu,\alpha\beta}, \quad (C.23)$$

$$\begin{aligned} k^{\rho_1} I_{\mu\nu,\rho_1\sigma} k^{\rho_2} I_{\alpha\beta,\rho_2\sigma} &= \frac{1}{4}(k_\mu k_\alpha g_{\nu\beta} - k_\mu k_\beta g_{\nu\alpha} - k_\nu k_\alpha g_{\mu\beta} + k_\nu k_\beta g_{\mu\alpha}) \\ &= \frac{1}{2}k^2(I - P)_{\mu\nu,\alpha\beta}, \end{aligned} \quad (C.24)$$

where we have introduced

$$P_{\mu\nu,\alpha\beta} := \frac{1}{2}(T_{\mu\alpha}T_{\nu\beta} - T_{\mu\beta}T_{\nu\alpha}), \quad T_{\mu\nu} := g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}. \quad (\text{C.25})$$

Using the Laurent expansion of the Gamma function $\Gamma(x)$, we obtain

$$\begin{aligned} & \Pi_{\mu\nu,\alpha\beta}^{ij}(k) \\ = & \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} I_{\mu\nu,\alpha\beta} \epsilon^{-1} \int_0^1 dx [\alpha + (1-\alpha)x] \\ & + \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} I_{\mu\nu,\alpha\beta} \int_0^1 dx \left\{ \left[-x(1-x) \frac{k^2}{M^2} \right] \ln \left[\frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \right. \\ & - \left[\alpha + (1-\alpha)x - x(1-x) \frac{k^2}{M^2} \right] \ln \left[\left\{ \alpha + (1-\alpha)x \right\} \frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \\ & + (\gamma_E - \ln 4\pi - 1)(1-\alpha)(1-x) - \gamma_E + \ln 4\pi \Big\} \\ & + \frac{-2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} \frac{1}{2} k^2 (I - P)_{\mu\nu,\alpha\beta} \\ & \times \left[M^{-2} \int_0^1 dx (2x^2 + 2x + 1) \left\{ \ln \left[\frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \right. \right. \\ & \quad \left. \left. - \ln \left[\left\{ \alpha + (1-\alpha)x \right\} \frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \right\} \right. \\ & - \frac{1}{2} M^{-2} \int_0^1 dx \left[1 - x(1-x) \frac{k^2}{M^2} \right] \ln \left[\frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \\ & - \frac{1}{2} M^{-2} \int_0^1 dx \left[\alpha - x(1-x) \frac{k^2}{M^2} \right] \ln \left[\alpha \frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \\ & + M^{-2} \int_0^1 dx \left[\alpha + (1-\alpha)x - x(1-x) \frac{k^2}{M^2} \right] \\ & \left. \times \ln \left[\left\{ \alpha + (1-\alpha)x \right\} \frac{M^2}{\mu^2} - x(1-x) \frac{k^2}{\mu^2} \right] \right] + O(\epsilon). \end{aligned} \quad (\text{C.27})$$

Here note that the divergent term $\epsilon^{-1}k^2(1-P)$ does not exist. In the neighborhood of $k^2 = 0$, i.e, in the low-energy region such that $k^2/M^2 \ll 1$,

$$\begin{aligned} & \Pi_{\mu\nu,\alpha\beta}^{ij}(k) \\ = & -\frac{2C_2\sigma^2 g^2}{16\pi^2} \delta^{ij} \left\{ I_{\mu\nu,\alpha\beta} \left[\epsilon^{-1} \frac{1+\alpha}{2} + f_0(\alpha) + f_1(\alpha) \frac{k^2}{M^2} + f_2(\alpha) \frac{k^4}{M^4} \right] \right. \\ & \left. + \frac{1}{2} \frac{k^2}{M^2} (I - P)_{\mu\nu,\alpha\beta} \left[h_0(\alpha) + h_1(\alpha) \frac{k^2}{M^2} \right] \right\} + O\left(\frac{k^6}{M^6}\right), \end{aligned} \quad (\text{C.28})$$

where²⁴

$$f_0(\alpha) := -\frac{1+\alpha}{2} \ln \frac{M^2}{\mu^2} + (\gamma_E - \ln 4\pi - 1)(1-\alpha) \frac{1}{2} - \gamma_E + \ln 4\pi$$

²⁴ The piece proportional to $1 - P$ leads to the perimeter law. Therefore, it is neglected when we obtain the string tension.

$$-\int_0^1 dx [\alpha + (1-\alpha)x] \ln[\alpha + (1-\alpha)x] \quad (\text{C.29})$$

$$= -\frac{1+\alpha}{2} \ln \frac{M^2}{\mu^2} + (\gamma_E - \ln 4\pi - 1)(1-\alpha)\frac{1}{2} - \gamma_E + \ln 4\pi$$

$$+ \frac{1}{1-\alpha} \left[\frac{\alpha^2}{2} \ln \alpha + \frac{1}{4} - \frac{1}{4}\alpha^2 \right], \quad (\text{C.30})$$

$$f_1(\alpha) := \int_0^1 dx x(1-x) \{1 + \ln[\alpha + (1-\alpha)x]\} \quad (\text{C.31})$$

$$= \frac{1}{6} + \frac{1}{(1-\alpha)^3} \left\{ \frac{\alpha^3}{3} \ln \alpha - \frac{\alpha^3}{9} + \frac{1}{9} - (1+\alpha) \left[\frac{\alpha^2}{2} \ln \alpha - \frac{\alpha^2}{4} + \frac{1}{4} \right] \right.$$

$$\left. + \alpha(\alpha \ln \alpha - \alpha + 1) \right\}, \quad (\text{C.32})$$

$$f_2(\alpha) := \int_0^1 dx \left[x^2(1-x)^2 - \frac{1}{2} \frac{x^2(1-x)^2}{\alpha + (1-\alpha)x} \right] \quad (\text{C.33})$$

$$= \frac{1}{30} - \frac{1}{(1-\alpha)^5} \left[\frac{1}{24} - \frac{1}{3}\alpha + \frac{1}{3}\alpha^3 - \frac{1}{24}\alpha^4 - \frac{1}{2}\alpha^2 \ln \alpha \right], \quad (\text{C.34})$$

and

$$h_0(\alpha) := \int_0^1 dx \left\{ -\frac{1}{2}\alpha \ln \alpha + [\alpha + (1-\alpha)x] \ln[\alpha + (1-\alpha)x] \right.$$

$$\left. - (2x^2 + 2x + 1) \ln[\alpha + (1-\alpha)x] \right\}, \quad (\text{C.35})$$

$$h_1(\alpha) := \int_0^1 dx \left\{ (2x^2 + 2x + 1) \frac{(1-\alpha)x(1-x)^2}{\alpha + (1-\alpha)x} \right.$$

$$\left. + \frac{1}{2}x(1-x) \ln \alpha - x(1-x) \ln[\alpha + (1-\alpha)x] \right\}. \quad (\text{C.36})$$

For $\alpha = 1$, the results are greatly simplified; $h_i = 0$ and $f_0 = -\ln \frac{M^2}{\mu^2} - \gamma_E + \ln 4\pi$, $f_1 = 1/6$, $f_2 = 1/60$.

D Manifest covariant quantization of the second rank antisymmetric tensor gauge field

D.1 Second-rank antisymmetric tensor gauge theory

We discuss the gauge fixing of a second-rank antisymmetric tensor gauge field $A_{\mu\nu}$ whose Lagrangian is given by

$$\mathcal{L}_0 = -\frac{1}{8} (\epsilon_{\mu\nu\rho\sigma} \partial^\nu A^{\rho\sigma})^2. \quad (\text{D.1})$$

This Lagrangian is invariant under the hypergauge transformation,

$$\delta A_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x). \quad (\text{D.2})$$

In order to fix the gauge, we adopt the gauge fixing condition for $A_{\mu\nu}$,

$$\partial^\nu A_{\mu\nu} = 0. \quad (\text{D.3})$$

Then the gauge fixing (GF) and Faddeev-Popov (FP) ghost term is obtained based on the prescription of Kugo and Uehara [73] as

$$\mathcal{L}_1 := -i\delta_B \left[\bar{C}^\nu (\partial^\mu A_{\mu\nu} + \frac{\alpha_1}{2} B_\nu) \right], \quad (\text{D.4})$$

where we have defined the nilpotent BRST transformation,

$$\begin{aligned} \delta_B A_{\mu\nu}(x) &= \partial_\mu C_\nu(x) - \partial_\nu C_\mu(x), \\ \delta_B C_\mu(x) &= i\partial_\mu d(x), \\ \delta_B d(x) &= 0, \\ \delta_B \bar{C}_\mu(x) &= iB_\mu(x), \\ \delta_B B_\mu(x) &= 0. \end{aligned} \quad (\text{D.5})$$

Hence, the explicit form of \mathcal{L}_1 reads

$$\mathcal{L}_1 = B^\nu \partial^\mu A_{\mu\nu} + i\bar{C}^\nu \partial^\mu [\partial_\mu C_\nu - \partial_\nu C_\mu] + \frac{\alpha_1}{2} (B_\mu)^2. \quad (\text{D.6})$$

However, the Lagrangian \mathcal{L}_1 and hence $\mathcal{L}_0 + \mathcal{L}_1$ is still invariant under the transformation of the vector ghosts C_μ and \bar{C}_μ , i.e., $\delta C_\mu(x) = i\partial_\mu \theta(x)$, $\delta \bar{C}_\mu(x) = i\partial_\mu \varphi(x)$. Note that C_μ and \bar{C}_μ are independent fields and that $C^\dagger = C$, $\bar{C}^\dagger = \bar{C}$. We consider the gauge fixing conditions, $\partial^\mu \bar{C}_\mu = 0$ and $\partial^\mu C_\mu = 0$. Therefore, we must add an additional GF+FP term,

$$\mathcal{L}_2 := -i\delta_B \left[\bar{d} (\partial^\mu C_\mu + \alpha_2 P) \right] - i\delta_B \left[N (\partial^\mu \bar{C}_\mu + \alpha_3 B^{(1)}) \right]. \quad (\text{D.7})$$

where the nilpotent BRST transformation of the additional fields is supplemented as

$$\begin{aligned} \delta_B N(x) &= P(x), \\ \delta_B P(x) &= 0, \\ \delta_B \bar{d}(x) &= B^{(1)}(x), \\ \delta_B B^{(1)}(x) &= 0, \end{aligned} \quad (\text{D.8})$$

The explicit form of \mathcal{L}_2 reads

$$\mathcal{L}_2 = -iB^{(1)} \partial^\mu C_\mu - i\alpha_4 B^{(1)} P + \bar{d} \partial^\mu \partial_\mu d - iP \partial^\mu \bar{C}_\mu + N \partial^\mu B_\mu, \quad (\text{D.9})$$

where we have defined $\alpha_4 := \alpha_2 - \alpha_3$. Note that P and $B^{(1)}$ anti-commute. For the assignment of the ghost number of each field, see Table.1. Two vector fields C_μ, \bar{C}_μ are two primary ghosts, and three scalar fields d, \bar{d}, N are three secondary ghosts. Three fields $B_\mu, P, B^{(1)}$ are the Lagrange multiplier fields for the condition, $\partial^\nu A_{\mu\nu} = 0, \partial^\mu \bar{C}_\mu = 0, \partial^\mu C_\mu = 0$, respectively. Thus we obtain the GF+FP term for the Lagrangian \mathcal{L}_0 , i.e., $\mathcal{L}_{GF+FP} = \mathcal{L}_1 + \mathcal{L}_2$.

field	rank	ghost number
A	2	0
C	1	1
d	0	2
\bar{C}	1	-1
B	1	0
N	0	0
P	0	1
\bar{d}	0	-2
$B^{(1)}$	0	-1
Λ	1	0
C'	0	1
\bar{C}'	0	-1
B'	0	0

Table 1: The ghost number of the field with the indicated rank.

Now all the gauge freedom is fixed. Thus the full Lagrangian density is given by

$$\begin{aligned}
\mathcal{L}_{tot} &= \mathcal{L}_0 + \mathcal{L}_{GF+FP}, \\
&= \mathcal{L}_0 + B^\nu \partial^\mu A_{\mu\nu} + i\bar{C}^\nu \partial^\mu [\partial_\mu C_\nu - \partial_\nu C_\mu] + \frac{\alpha_1}{2}(B_\mu)^2 \\
&\quad - iB^{(1)} \partial^\mu C_\mu - i\alpha_4 B^{(1)} P + \bar{d} \partial^\mu \partial_\mu d - iP \partial^\mu \bar{C}_\mu + N \partial^\mu B_\mu. \quad (D.10)
\end{aligned}$$

The massless antisymmetric tensor field stands for the massless spin-0 field as a physical mode. It is possible to show that all the unphysical modes decouple leaving correctly one physical mode [39].

The above result is the summary of the results obtained by Townsend [38], Hata, Kugo and Ohta [40] and Kimura [39]. The same result can be obtained within the framework of the extended theory for the constrained system based on the canonical Hamiltonian formalism on the extended phase space, the so-called the Batalin-Fradkin-Vilkovisky (BFV) formalism, see e.g. the original papers and a review [74].

D.2 Inclusion of mass term

Next, we consider the theory of an antisymmetric tensor field with the mass term,

$$\mathcal{L}_0^m[A] = -\frac{1}{8}(\epsilon_{\mu\nu\rho\sigma} \partial^\nu A^{\rho\sigma})^2 - \frac{1}{4}m^2(A_{\mu\nu})^2. \quad (D.11)$$

This Lagrangian with the mass term is no longer invariant under the hypergauge transformation of $A_{\mu\nu}$. However, the invariance is recovered by introducing an additional vector field Λ_μ in such a way that

$$\mathcal{L}_0^{m'}[A, \Lambda] = -\frac{1}{8}(\epsilon_{\mu\nu\rho\sigma} \partial^\nu A^{\rho\sigma})^2 - \frac{1}{4}(mA_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu)^2. \quad (D.12)$$

Actually, this Lagrangian is invariant under the combined transformation,

$$\delta A_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) - \partial_\nu \xi_\mu(x), \quad \delta \Lambda_\mu(x) = -m \xi_\mu(x). \quad (\text{D.13})$$

Moreover, it has an additional invariance under the transformation,

$$\delta A_{\mu\nu}(x) = 0, \quad \delta \Lambda_\mu(x) = \partial_\mu \omega(x). \quad (\text{D.14})$$

Therefore, we define the BRST transformation of the field Λ as

$$\delta_B \Lambda_\mu(x) = -m C_\mu(x) + \partial_\mu C'(x), \quad (\text{D.15})$$

by introducing additional ghost C' . The BRST transformation of C' and \bar{C}' is defined by

$$\delta_B C'(x) = i m d(x), \quad (\text{D.16})$$

$$\delta_B \bar{C}'(x) = i B'(x), \quad (\text{D.17})$$

$$\delta_B B'(x) = 0. \quad (\text{D.18})$$

The first BRST transformation is determined by nilpotency of (D.15). Other two transformations are also suggested from nilpotency of BRST transformation. Thus the nilpotent BRST transformation is determined for all the fields. In order to fix the gauge freedom, we must obtain the GF+PF term \mathcal{L}_{GF+FP} . For $\mathcal{L}'_{tot} := \mathcal{L}_0^{m'} + \mathcal{L}_{GF+FP}$, the generation functional of the theory is given by

$$Z := \int \mathcal{D}A_{\mu\nu} \mathcal{D}\Lambda_\mu \mathcal{D}B_\mu \mathcal{D}C_\mu \mathcal{D}\bar{C}_\mu \mathcal{D}d \mathcal{D}\bar{d} \mathcal{D}N \mathcal{D}P \mathcal{D}B^{(1)} \mathcal{D}C' \mathcal{D}\bar{C}' \mathcal{D}B' \exp \left[i \int d^4x \mathcal{L}'_{tot} \right]. \quad (\text{D.19})$$

A good choice is²⁵

$$\begin{aligned} \mathcal{L}_{GF+FP} &= -i\delta_B \left[\bar{C}^\nu \left(\partial^\mu A_{\mu\nu} - \partial_\nu N - a\Lambda_\nu + \frac{\alpha_1}{2} B_\nu \right) + \bar{d} (\partial^\mu C_\mu + bC' + \alpha_2 P) \right. \\ &\quad \left. + \bar{C}' \left(\partial^\mu \Lambda_\mu + \frac{\alpha'}{2} B' \right) \right] \\ &= B^\nu \left(\partial^\mu A_{\mu\nu} - \partial_\nu N - a\Lambda_\nu + \frac{\alpha_1}{2} B_\nu \right) \\ &\quad + i\bar{C}^\nu [\partial^\mu (\partial_\mu C_\nu - \partial_\nu C_\mu) - \partial_\nu P - a(\partial_\nu C' - mC_\nu)] \\ &\quad - iB^{(1)} (\partial^\mu C_\mu + bC' + \alpha_2 P) + \bar{d} \partial^\mu \partial_\mu d + b m \bar{d} d \\ &\quad + B' (\partial^\mu \Lambda_\mu + \frac{\alpha'}{2} B') + i\bar{C}' \partial^\mu (\partial_\mu C' - mC_\mu), \end{aligned} \quad (\text{D.20})$$

where $\alpha_1, \alpha_2, \alpha'$ are gauge fixing parameters and a, b are parameters specified later. From the naive viewpoint, this corresponds to the gauge fixing condition, $\partial^\mu A_{\mu\nu} = 0$, $\partial^\mu C_\mu = 0 = \partial^\mu \bar{C}_\mu$ and $\partial^\mu \Lambda_\mu = 0$. If we integrate over B and B' , the sector containing A and Λ reads

$$Z := \int \mathcal{D}A_{\mu\nu} \mathcal{D}\Lambda_\mu \mathcal{D}N \exp \left[i \int d^4x \mathcal{L}''_{tot} \right], \quad (\text{D.22})$$

²⁵ The author would like to thank Atsushi Nakamura [75] for helpful discussions on this Appendix.

where

$$\begin{aligned}\mathcal{L}_{tot}'' &= \mathcal{L}_0^m[A] - \frac{1}{2}mA_{\mu\nu}(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu) - \frac{1}{4}m^2(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu)^2 \\ &\quad - \frac{1}{2\alpha'}(\partial^\mu\Lambda_\mu)^2 + \frac{1}{2\alpha_1}(\partial^\nu A_{\mu\nu} - \partial_\mu N - a\Lambda_\mu)^2,\end{aligned}\quad (\text{D.23})$$

since other fields decouple from the relevant sector. Then we perform the integration over N and obtain

$$\begin{aligned}Z &= \int \mathcal{D}A_{\mu\nu} \exp \left\{ i \int d^4x \left(\mathcal{L}_0^m[A] + \frac{1}{2\alpha_1} (\partial^\nu A_{\mu\nu})^2 \right) \right\} \\ &\quad \times \int \mathcal{D}\Lambda_\mu \exp \left\{ i \int d^4x \mathcal{L}_1^m[A, \Lambda] \right\},\end{aligned}\quad (\text{D.24})$$

where

$$\begin{aligned}\mathcal{L}_1^m[A, \Lambda] &:= -\frac{1}{2}mA_{\mu\nu}(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu) - \frac{1}{4}m^2(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu)^2 \\ &\quad + \frac{a^2}{2\alpha_1}(\Lambda_\mu)^2 + \frac{a}{\alpha_1}(\partial^\nu A_{\mu\nu})\Lambda_\mu + \frac{a^2}{2\alpha_1}(\partial^\mu\Lambda_\mu)\Delta^{-1}(\partial^\nu\Lambda_\nu).\end{aligned}\quad (\text{D.25})$$

If we choose $a = m\alpha_1$, the cross term $(\partial^\nu A_{\mu\nu})\Lambda_\mu$ cancels with $mA_{\mu\nu}(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu)$. Hence, $\mathcal{L}_1^m[A, \Lambda]$ becomes independent of A field. Thus the Λ field decouples from the theory of A ,

$$\mathcal{L}_1^m[\Lambda] = -\frac{1}{4}m^2(\partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu)^2 + \frac{a^2}{2\alpha_1}(\Lambda_\mu)^2 + \frac{a^2}{2\alpha_1}(\partial^\mu\Lambda_\mu)\Delta^{-1}(\partial^\nu\Lambda_\nu).\quad (\text{D.26})$$

By taking the Landau gauge $\alpha_1 = 0$, we recover the partition function,

$$Z = \int \mathcal{D}A_{\mu\nu} \delta(\partial^\nu A_{\mu\nu}) \exp \left\{ i \int d^4x \mathcal{L}_0^m[A] \right\}.\quad (\text{D.27})$$

Moreover, it is possible to consider a simpler gauge [76],

$$\mathcal{L}_{GF+FP} = -i\delta_B \left[\bar{C}^\mu \Lambda_\mu + \bar{d}C' \right]\quad (\text{D.28})$$

$$= B^\mu \Lambda_\mu + i\bar{C}^\mu (\partial_\mu C' - mC_\mu) - iB^{(1)}C' + m\bar{d}d.\quad (\text{D.29})$$

This corresponds to the gauge fixing condition, $\Lambda_\mu = 0$ and $C' = 0$. The integration over N , P and B' is trivial, since \mathcal{L}'_{tot} does not include them. The B integration leads to the constraint $\delta(\Lambda_\mu)$,

$$Z = \int \mathcal{D}A_{\mu\nu} \mathcal{D}\Lambda_\mu \delta(\Lambda_\mu) \int \mathcal{D}C_\mu \mathcal{D}\bar{C}_\mu \mathcal{D}d \mathcal{D}\bar{d} \mathcal{D}B^{(1)} \mathcal{D}C' \mathcal{D}\bar{C}' \exp \left[i \int d^4x \mathcal{L}_{tot}'' \right],\quad (\text{D.30})$$

where

$$\mathcal{L}_{tot}'' = \mathcal{L}_0^{m'}[A_{\mu\nu}, \Lambda_\mu] + i\bar{C}^\mu (\partial_\mu C' - mC_\mu) - iB^{(1)}C' + m\bar{d}d.\quad (\text{D.31})$$

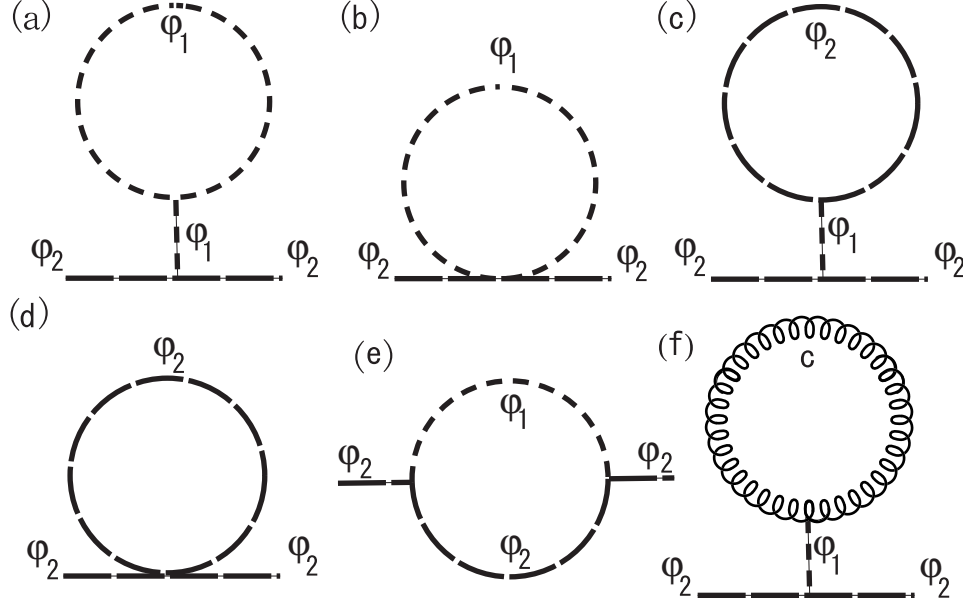


Figure 5: Self-energy diagrams for the would-be Nambu-Goldstone particle φ_2 .

When we consider the sector of A and Λ , the sector described by other fields decouples and we obtain

$$Z = \int \mathcal{D}A_{\mu\nu} \mathcal{D}\Lambda_\mu \delta(\Lambda_\mu) \exp \left\{ i \int d^4x \mathcal{L}_0^{m'}[A_{\mu\nu}, \Lambda_\mu] \right\} = \int \mathcal{D}A_{\mu\nu} \exp \left\{ i \int d^4x \mathcal{L}_0^m[A_{\mu\nu}] \right\}. \quad (\text{D.32})$$

Therefore, we recover the original theory which is written by the A field only with the Lagrangian \mathcal{L}_0^m . Note that the massive antisymmetric tensor gauge theory stands for the massive spin-1 theory.

E Renormalization of dual Abelian Higgs model

E.1 Goldstone bosons remain massless in higher orders

We examine the self-energy diagrams for the would-be NG boson which are given in Fig. 5.

$$\Sigma_a(0) := (-2i\lambda v) \frac{i}{-m^2} (-6i\lambda v) \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = -3\lambda I(m^2), \quad (\text{E.1})$$

$$\Sigma_b(0) := (-2i\lambda) \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} = \lambda I(m^2), \quad (\text{E.2})$$

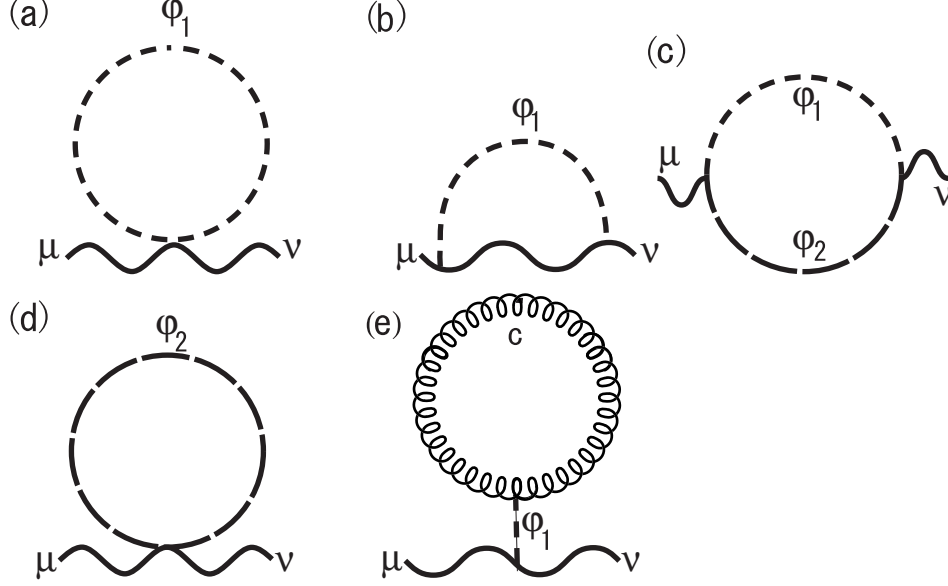


Figure 6: Vacuum polarization diagrams for the b_μ particle.

$$\Sigma_c(0) := (-2i\lambda v) \frac{i}{-m^2} (-2i\lambda v) \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2} = -\lambda I(\xi M^2), \quad (\text{E.3})$$

$$\Sigma_d(0) := (-6i\lambda) \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2} = 3\lambda I(\xi M^2), \quad (\text{E.4})$$

$$\begin{aligned} \Sigma_e(0) &:= (-2i\lambda v)^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2} \frac{i}{k^2 - m^2} \\ &= \frac{4\lambda^2}{2\lambda - \xi g^2} [I(m^2) - I(\xi M^2)], \end{aligned} \quad (\text{E.5})$$

$$\Sigma_f(0) := (-2i\lambda v) \frac{i}{-m^2} (ig\xi M) \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2} = \xi g^2 I(\xi M^2), \quad (\text{E.6})$$

When $\xi = 0$, it is easy to check that the total sum of the above self-energy parts vanishes. This fact ensures that the Nambu-Goldstone particle remains massless even if we include the higher order corrections.

E.2 Vacuum polarization

Next, we consider the vacuum polarization for the b_μ field. The relevant diagrams to one loop are enumerated in Fig. 6. The vacuum polarization tensor for the dual

vector field b_μ is obtained by summing up the following contributions,

$$\Pi_{\mu\nu}^{(a)}(p) := 2ig_m^2 g_{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2}, \quad m^2 := 2\mu^2 = 2\lambda v^2, \quad (\text{E.7})$$

$$\begin{aligned} \Pi_{\mu\nu}^{(b)}(p) &:= (2ig_m^2 v g_{\mu\rho})(2ig_m^2 v g_{\sigma\nu}) \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{-i}{k^2 - M^2} \\ &\quad \times \left[g_{\rho\sigma} - (1-\xi) \frac{k_\rho k_\sigma}{k^2 - \xi M^2} \right], \end{aligned} \quad (\text{E.8})$$

$$\Pi_{\mu\nu}^{(c)}(p) := g_m^2 \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{i}{k^2 - \xi M^2} (2k-p)_\mu (-2k+p)_\nu, \quad (\text{E.9})$$

$$\Pi_{\mu\nu}^{(d)}(p) := 2ig_m^2 g_{\mu\nu} \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2}, \quad (\text{E.10})$$

$$\Pi_{\mu\nu}^{(e)}(p) := 2ig_m^2 v g_{\mu\nu} i g \xi M \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \xi M^2} \frac{i}{-m^2}. \quad (\text{E.11})$$

First, the dimensional regularization yields

$$\Pi_{\mu\nu}^{(a)}(p) := -2g_m^2 g_{\mu\nu} \int \frac{d^D k}{i(2\pi)^D} \frac{1}{k^2 - m^2} = i2g_m^2 g_{\mu\nu} \int \frac{d^D k}{i(2\pi)^D} \frac{1}{-k^2 + m^2} \quad (\text{E.12})$$

$$= i2g_m^2 g_{\mu\nu} \frac{\Gamma(\epsilon-1)}{(4\pi)^{2-\epsilon}} (m^2)^{1-\epsilon} \quad (\text{E.13})$$

$$= i2g_m^2 g_{\mu\nu} \frac{1}{(4\pi)^2} \left(-\frac{1}{\epsilon} + \gamma_E - 1 \right) m^2 (1 - \epsilon \ln m^2) (1 + \epsilon \ln 4\pi) \quad (\text{E.14})$$

$$= i \frac{2g_m^2}{(4\pi)^2} g_{\mu\nu} m^2 \left(-N_\epsilon - 1 + \ln m^2 \right), \quad (\text{E.15})$$

where we have defined $\epsilon := 2 - D/2$ and

$$N_\epsilon := \frac{1}{\epsilon} + \ln 4\pi - \gamma_E. \quad (\text{E.16})$$

and used the Laurent expansion of the Gamma function $\Gamma(\epsilon-1)$.

Second, we want to calculate

$$\begin{aligned} &\Pi_{\mu\nu}^{(b)}(p) \\ &:= (2ig_m^2 v g_{\mu\rho})(2ig_m^2 v g_{\sigma\nu}) \int \frac{d^4 k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2} \frac{-i}{k^2 - M^2} \left[g_{\rho\sigma} - (1-\xi) \frac{k_\rho k_\sigma}{k^2 - \xi M^2} \right] \\ &= -4g_m^4 v^2 \int \frac{d^4 k}{(2\pi)^4} \frac{1}{-(p-k)^2 + m^2} \left[\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}}{-k^2 + M^2} + \frac{\frac{k_\mu k_\nu}{M^2}}{-k^2 + \xi M^2} \right]. \end{aligned} \quad (\text{E.17})$$

The Feynman parameter formula can combine two denominators into a common denominator. Then the dimensional regularization leads to

$$\Pi_{\mu\nu}^{(b)}(p)$$

$$\begin{aligned}
&= -i4g_m^4 v^2 \int_0^1 dx \left\{ \int \frac{d^D k}{i(2\pi)^D} \frac{\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)}{[-k^2 + 2xp \cdot k - xp^2 + xm^2 + (1-x)M^2]^2} \right. \\
&\quad \left. + \int \frac{d^D k}{i(2\pi)^D} \frac{\frac{k_\mu k_\nu}{M^2}}{[-k^2 + 2xp \cdot k - xp^2 + xm^2 + (1-x)\xi M^2]^2} \right\} \quad (E.18)
\end{aligned}$$

$$\begin{aligned}
&= \frac{-i4g_m^4 v^2}{(4\pi)^{2-\epsilon} \Gamma(2)} \int_0^1 dx \left\{ \Gamma(\epsilon) g_{\mu\nu} [(x^2 - x)p^2 + xm^2 + (1-x)M^2]^{-\epsilon} \right. \\
&\quad - \Gamma(\epsilon) x^2 \frac{p_\mu p_\nu}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)M^2]^{-\epsilon} \\
&\quad + \Gamma(\epsilon - 1) \frac{1}{2} \frac{g_{\mu\nu}}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)M^2]^{1-\epsilon} \\
&\quad + \Gamma(\epsilon) x^2 \frac{p_\mu p_\nu}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2]^{-\epsilon} \\
&\quad \left. - \Gamma(\epsilon - 1) \frac{1}{2} \frac{g_{\mu\nu}}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2]^{1-\epsilon} \right\}. \quad (E.19)
\end{aligned}$$

Hence, using the Laurent expansion of the Gamma function $\Gamma(\epsilon)$, we have

$$\begin{aligned}
&\Pi_{\mu\nu}^{(b)}(p) \\
&= \frac{-i4g_m^4 v^2}{(4\pi)^2 \Gamma(2)} \int_0^1 dx \left[g_{\mu\nu} \left\{ N_\epsilon - \ln[(x^2 - x)p^2 + xm^2 + (1-x)M^2] \right\} \right. \\
&\quad + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)M^2] \\
&\quad \times \left\{ -N_\epsilon - 1 + \ln[(x^2 - x)p^2 + xm^2 + (1-x)M^2] \right\} \\
&\quad + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} [(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2] \\
&\quad \times \left\{ N_\epsilon + 1 - \ln[(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2] \right\} \\
&\quad + x^2 \frac{p_\mu p_\nu}{M^2} \left\{ + \ln[(x^2 - x)p^2 + xm^2 + (1-x)M^2] \right. \\
&\quad \left. \left. - \ln[(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2] \right\} \right] + O(\epsilon). \quad (E.20)
\end{aligned}$$

Using the low-energy expansion,

$$\begin{aligned}
&\ln[(x^2 - x)p^2 + xm^2 + (1-x)\xi M^2] \\
&= \ln[xm^2 + (1-x)\xi M^2] + \frac{x^2 - x}{xm^2 + (1-x)\xi M^2} p^2 + O\left(\frac{p^4}{M^4}\right), \quad (E.21)
\end{aligned}$$

we obtain

$$\begin{aligned}
&\Pi_{\mu\nu}^{(b)}(p) \\
&= \frac{-i4g_m^4 v^2}{(4\pi)^2 \Gamma(2)} \left[g_{\mu\nu} \int_0^1 dx \left\{ N_\epsilon - \ln[xm^2 + (1-x)M^2] + \frac{x^2 - x}{xm^2 + (1-x)M^2} p^2 \right\} \right. \\
&\quad \left. + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} \int_0^1 dx [xm^2 + (1-x)M^2] \right]
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ -N_\epsilon - 1 + \ln[xm^2 + (1-x)M^2] + \frac{x^2 - x}{xm^2 + (1-x)M^2} p^2 \right\} \\
& + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} p^2 \int_0^1 dx (x^2 - x) \left\{ -N_\epsilon - 1 + \ln[xm^2 + (1-x)M^2] \right\} \\
& + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} \int_0^1 dx [xm^2 + (1-x)\xi M^2] \\
& \times \left\{ N_\epsilon + 1 - \ln[xm^2 + (1-x)\xi M^2] - \frac{x^2 - x}{xm^2 + (1-x)\xi M^2} p^2 \right\} \\
& + \frac{1}{2} \frac{g_{\mu\nu}}{M^2} p^2 \int_0^1 dx (x^2 - x) \left\{ N_\epsilon + 1 - \ln[xm^2 + (1-x)\xi M^2] \right\} \\
& + \frac{p_\mu p_\nu}{M^2} \int_0^1 dx x^2 \left\{ + \ln[xm^2 + (1-x)M^2] - \ln[xm^2 + (1-x)\xi M^2] \right\} \Bigg] \\
& + O(p^4). \tag{E.22}
\end{aligned}$$

In particular, when $\xi = 0$, we obtain

$$\begin{aligned}
\Pi_{\mu\nu}^{(b)}(p) &= -i \frac{4g_m^4 v^2}{(4\pi)^2} g_{\mu\nu} \left[\frac{3}{4}(N+1) - \frac{m^2 \ln m^2 - M^2 \ln M^2}{m^2 - M^2} - \frac{1}{8} \frac{m^2}{M^2} (-1 + 2 \ln m^2) \right] \\
&+ O\left(\frac{p^2}{M^2}\right). \tag{E.23}
\end{aligned}$$

Third, we can proceed in the similar way to (b) and obtain

$$\begin{aligned}
& \Pi_{\mu\nu}^{(c)}(p) \tag{E.24} \\
&= i \frac{g_m^2}{(4\pi)^2} \left[g_{\mu\nu} (N_\epsilon + 1) \left(-\frac{1}{3} p^2 + m^2 - \xi M^2 \right) \right. \\
&\quad - 2g_{\mu\nu} \left\{ \int_0^1 dx [xm^2 + (1-x)\xi M^2] \ln[xm^2 + (1-x)\xi M^2] \right. \\
&\quad \left. + p^2 \int_0^1 dx (x^2 - x) + p^2 \int_0^1 dx (x^2 - x) \ln[xm^2 + (1-x)\xi M^2] \right\} \\
&\quad \left. + \frac{1}{3} N_\epsilon p_\mu p_\nu - p_\mu p_\nu \int_0^1 dx (4x^2 - 4x + 1) \ln[xm^2 + (1-x)\xi M^2] \right] + O(p^4). \tag{E.25}
\end{aligned}$$

In particular, when $\xi = 0$, we obtain

$$\begin{aligned}
\Pi_{\mu\nu}^{(c)}(p) &= i \frac{g_m^2}{(4\pi)^2} \left\{ g_{\mu\nu} \left[p^2 \left(-\frac{N_\epsilon}{3} - \frac{13}{36} - 6 \ln m^2 \right) + m^2 \left(N_\epsilon + \frac{3}{2} - \ln m^2 \right) \right] \right. \\
&\quad \left. + p_\mu p_\nu \left[\frac{N_\epsilon}{3} + \frac{4}{9} - \frac{1}{3} \ln m^2 \right] \right\} + O(p^4). \tag{E.26}
\end{aligned}$$

The remaining quantities, $\Pi_{\mu\nu}^{(d)}$ and $\Pi_{\mu\nu}^{(e)}$ are calculated in the same manner as $\Pi_{\mu\nu}^{(a)}$. However, they vanish for $\xi = 0$ in the dimensional regularization.

Therefore, the total sum of the vacuum polarization for $\xi = 0$ is given by

$$\begin{aligned}
& i\Pi_{\mu\nu}(p) \\
= & -\frac{2g_m^2}{(4\pi)^2}m^2g_{\mu\nu}\left(-N_\epsilon - 1 + \ln \frac{m^2}{\mu^2}\right) \\
& + \frac{4g_m^2}{(4\pi)^2}(g_mv)^2g_{\mu\nu}\left[\frac{3}{4}(N_\epsilon + 1) - \frac{m^2 \ln \frac{m^2}{\mu^2} - M^2 \ln \frac{M^2}{\mu^2}}{m^2 - M^2} - \frac{1}{8}\frac{m^2}{M^2}\left(-1 + 2 \ln \frac{m^2}{\mu^2}\right)\right] \\
& - \frac{g_m^2}{(4\pi)^2}m^2g_{\mu\nu}\left(N_\epsilon + \frac{3}{2} - \ln \frac{m^2}{\mu^2}\right) \\
& - \frac{g_m^2}{(4\pi)^2}\left\{-p^2g_{\mu\nu}\left(\frac{N_\epsilon}{3} + \frac{13}{36} + 6 \ln \frac{m^2}{\mu^2}\right) + p_\mu p_\nu\left[\frac{N_\epsilon}{3} + \frac{4}{9} - \frac{1}{3} \ln \frac{m^2}{\mu^2}\right]\right\} \\
& + O\left(\frac{p^2}{M^2}\right), \tag{E.27}
\end{aligned}$$

where $m^2 = 2\lambda v^2$ and $M^2 = g_m^2 v^2$.

Finally, the renormalization factors are obtained as

$$\begin{aligned}
\delta_b &= Z_b - 1 = -\frac{g_m^2}{(4\pi)^2}\left(\frac{N_\epsilon}{3} + \frac{13}{36} + 6 \ln \frac{m^2}{\mu^2}\right) + O(g_m^4), \\
\delta_m &= Z_b(M_b)^2 - (M_b^R)^2 = -\frac{2g_m^2}{(4\pi)^2}m^2\left(-N_\epsilon - 1 + \ln \frac{m^2}{\mu^2}\right) \\
&+ \frac{4g_m^2}{(4\pi)^2}(g_mv)^2\left[\frac{3}{4}(N_\epsilon + 1) - \frac{m^2 \ln \frac{m^2}{\mu^2} - M^2 \ln \frac{M^2}{\mu^2}}{m^2 - M^2} - \frac{1}{8}\frac{m^2}{M^2}\left(-1 + 2 \ln \frac{m^2}{\mu^2}\right)\right] \\
&- \frac{g_m^2}{(4\pi)^2}m^2\left(N_\epsilon + \frac{3}{2} - \ln \frac{m^2}{\mu^2}\right) + O(g_m^4). \tag{E.28}
\end{aligned}$$

F Calculation of the Wilson loop

In the following we show that the area law decay of the Wilson loop is obtained from the result,

$$\langle W(C) \rangle_{YM} = \exp \left\{ -\frac{1}{2}(2Jg\rho^{-1}K^{1/2})^2(\tilde{\Xi}_\mu, D_m^{-1}\tilde{\Xi}^\mu) \right\}, \tag{F.1}$$

where Ξ is the one-form defined by

$$\Xi := *d\Theta\Delta^{-1} = \delta * \Theta\Delta^{-1}, \tag{F.2}$$

with the component,

$$\Xi^\mu(x) = \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\partial_\alpha^x \int d^4y \Theta_{\beta\gamma}(y)\Delta^{-1}(y-x) \tag{F.3}$$

$$= \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\partial_\alpha^x \int_S d^2S_{\beta\gamma}(x')\Delta^{-1}(x'-x). \tag{F.4}$$

Then the argument of the exponential is cast into the following form,

$$\begin{aligned}
(\Xi_\mu, D_m^{-1} \Xi^\mu) &= (\delta * \Theta \Delta^{-1}, D_m^{-1} \delta * \Theta \Delta^{-1}) \\
&= (\Theta, \Delta^{-1} * d\delta * d\Delta^{-1} D_m^{-1} \Theta) \\
&= (\Theta, \Delta^{-1} \delta d\Delta^{-1} D_m^{-1} \Theta) \\
&= (\Theta, \Delta^{-1} D_m^{-1}(\Delta) \Theta) - (\Theta, \Delta^{-1} d\delta \Delta^{-1} D_m^{-1} \Theta) \\
&= (\Theta, \Delta^{-1} D_m^{-1}(\Delta) \Theta) - (\delta \Theta, \Delta^{-2} D_m^{-1} \delta \Theta).
\end{aligned} \tag{F.5}$$

For the rectangular loop with side lengths T and R in the $x_1 - x_4$ plane, we take

$$\Theta_{\alpha\beta}(z) = \delta_{\alpha 1} \delta_{\beta 4} \delta(z_2) \delta(z_3) \theta(z_1) \theta(R - z_1) \theta(z_4) \theta(T - z_4). \tag{F.6}$$

Then the Fourier transformation is given by

$$\begin{aligned}
\Theta_{\alpha\beta}(p) &\equiv \int d^4 z \Theta_{\alpha\beta}(z) e^{-ip \cdot z} \\
&= \delta_{\alpha 1} \delta_{\beta 4} \int_0^R dz_1 e^{-ip_1 z_1} \int_0^T dz_4 e^{-ip_4 z_4} \\
&= \delta_{\alpha 1} \delta_{\beta 4} \frac{2}{p_1} e^{-i\frac{p_1 R}{2}} \sin \frac{p_1 R}{2} \frac{2}{p_4} e^{-i\frac{p_4 T}{2}} \sin \frac{p_4 T}{2}.
\end{aligned} \tag{F.7}$$

In the momentum representation, we have

$$(\Theta, \Delta^{-1} D_m^{-1} \Theta) = \int \frac{d^4 p}{(2\pi)^4} \Theta_{\alpha\beta}(p) \Theta_{\alpha\beta}(-p) [\Delta^{-1} D_m^{-1}](p). \tag{F.8}$$

If we use the formula following [24],

$$\lim_{R \rightarrow \infty} \left(\frac{\sin aR}{a} \right)^2 = \pi R \delta(a), \tag{F.9}$$

for large R and large T , then we obtain

$$\begin{aligned}
(\Theta, \Delta^{-1} D_m^{-1} \Theta) &\cong \int \frac{d^4 p}{(2\pi)^4} (2\pi)^2 T R \delta(p_1) \delta(p_4) [\Delta^{-1} D_m^{-1}](p) \\
&= T R \int \frac{d^2 p}{(2\pi)^2} [\Delta^{-1} D_m^{-1}](0, p_2, p_3, 0) \\
&= -T R \int \frac{d^2 p}{(2\pi)^2} \kappa \left[\frac{1}{p_2^2 + p_3^2 + m_1^2} - \frac{1}{p_2^2 + p_3^2 + m_2^2} \right].
\end{aligned} \tag{F.10}$$

Here the logarithmic divergence of the integral is removed by introducing the ultra-violet cutoff Λ as

$$\begin{aligned}
(\Theta, \Delta^{-1} D_m^{-1} \Theta) &= -T R \int \frac{d^2 p}{(2\pi)^2} \kappa \left[\frac{1}{p_2^2 + p_3^2 + m_1^2} - \frac{1}{p_2^2 + p_3^2 + m_2^2} \right] \\
&= -T R \lim_{\Lambda \rightarrow \infty} \int_0^{\Lambda^2} \frac{d|p|^2}{4\pi} \kappa \left[\frac{1}{|p|^2 + m_1^2} - \frac{1}{|p|^2 + m_2^2} \right] \\
&= T R \frac{\kappa}{4\pi} \lim_{\Lambda \rightarrow \infty} \ln \frac{m_1^2}{\Lambda^2 + m_1^2} \frac{\Lambda^2 + m_2^2}{m_2^2} \\
&= T R \frac{\kappa}{4\pi} \ln \frac{m_1^2}{m_2^2}.
\end{aligned} \tag{F.11}$$

So we obtain the Λ -independent result. The last term in (F.5) gives a perimeter decay part, since $\delta\Theta$ is the boundary of the rectangular surface. The dominant term in the large loop is given by the contribution (F.11) which exhibits the area law. Thus we arrive at the result (5.26).

G Derivative expansion of the string

In this section, we begin with the expression,

$$\langle W(C) \rangle_{YM} = \exp \left[- \int_{S_C} dS^{\mu\nu}(x) \int_{S_C} dS^{\rho\sigma}(y) G_{\mu\nu,\rho\sigma}(x, y) \right], \quad (\text{G.1})$$

where

$$G_{\mu\nu,\rho\sigma}(x, y) := -2J^2 g^2 \rho^{-2} K \langle \mathcal{D}[\partial_x] h_{\mu\nu}^\xi(x) \mathcal{D}[\partial_y] h_{\rho\sigma}^\xi(y) \rangle_{APEGT}. \quad (\text{G.2})$$

In this paper, we have obtained the result,

$$G_{\mu\nu,\rho\sigma}(x, y) = -2J^2 g^2 \rho^{-2} K I_{\mu\nu,\rho\sigma} [\Delta D_m]^{-1} \quad (\text{G.3})$$

$$= -2J^2 g^2 \rho^{-2} K I_{\mu\nu,\rho\sigma} \left[\frac{\chi}{\Delta - m_1^2} - \frac{\chi}{\Delta - m_2^2} \right]. \quad (\text{G.4})$$

We define the Euclidean propagator:

$$G_m(x) = (-\Delta_E + m^2)^{-1}(x, 0) = \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot x} \frac{1}{k^2 + m^2}. \quad (\text{G.5})$$

It is written in the form,

$$G_m(x) = \frac{1}{4\pi^2} \frac{m}{|x|} K_1(m|x|) = \frac{1}{4\pi^2} m^2 \frac{K_1(|x|/\xi)}{|x|/\xi}, \quad (\text{G.6})$$

where $K_1(z)$ is the modified Bessel function and we have defined the correlation length ξ by $\xi = m^{-1}$. (G.6) is obtained as follows. Substituting the identity,

$$\frac{1}{k^2 + m^2} = \int_0^\infty ds e^{-s(k^2 + m^2)}, \quad (\text{G.7})$$

into (G.5) and performing the Gaussian integration over the four momentum k , we obtain

$$\begin{aligned} G_m(x) &= \int_0^\infty ds e^{-sm^2} \int \frac{d^4 k}{(2\pi)^4} e^{-sk^2 + ik \cdot x} \\ &= \int_0^\infty ds e^{-sm^2} \exp \left[-\frac{x^2}{4s} \right] \frac{1}{(2\pi)^4} \left(\sqrt{\frac{\pi}{s}} \right)^4 \\ &= \frac{1}{16\pi^2} \int_0^\infty ds \frac{1}{s^2} \exp \left[-sm^2 - \frac{x^2}{4s} \right]. \end{aligned} \quad (\text{G.8})$$

The above result (G.6) is immediately obtained by applying the integration formula [71]:

$$\int_0^\infty dx x^{\nu^1} \exp \left[-\frac{\beta}{x} - \gamma x \right] = 2(\beta\gamma^{-1})^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \quad (\Re\beta > 0, \Re\gamma > 0), \quad (\text{G.9})$$

to the case $\nu = -1, \beta = x^2/4, \gamma = m^2$, since $K_{-\nu}(z) = K_\nu(z)$.

In Euclidean space,

$$G_{\mu\nu,\rho\sigma}(x, x') = -2J^2 g^2 \rho^{-2} K I_{\mu\nu,\rho\sigma} \kappa [G_{m_1}(x - x') - G_{m_2}(x - x')] \quad (\text{G.10})$$

$$= 2I_{\mu\nu,\rho\sigma} [F_1((x - x')^2) - F_2((x - x')^2)], \quad (\text{G.11})$$

$$F_i((x - x')^2) := -J^2 g^2 \rho^{-2} K \frac{\chi}{4\pi^2} \frac{m_i K_1(m_i |x - x'|)}{|x - x'|}. \quad (\text{G.12})$$

We define

$$J_i := 2 \int_{S_C} dS^{\mu\nu}(x(\sigma)) \int_{S_C} dS^{\rho\sigma}(x(\sigma')) I_{\mu\nu,\rho\sigma} F_i((x(\sigma) - x(\sigma'))^2). \quad (\text{G.13})$$

It is shown [65, 68] that the derivative expansion in powers of

$$\zeta^a := (\sigma' - \sigma)^a / \xi_i, \quad \xi_i := m_i^{-1}, \quad (\text{G.14})$$

leads to

$$J_i = \int d^2\sigma \sqrt{g} \left[4\xi_i^2 M_0^i - \frac{1}{4} \xi_i^4 M_2^i g^{ab} (\partial_a t_{\mu\nu}) (\partial_b t_{\mu\nu}) \right] + O(\xi_i^6), \quad (\text{G.15})$$

with the moment,

$$M_n^i := \int d^2z (z^2)^n F_i(z^2), \quad z^a := g^{1/4} \zeta^a, \quad (\text{G.16})$$

where we have used the conformal gauge for the induced metric, $g_{ab}(\sigma) = \sqrt{g(\sigma)} \delta_{ab}$, hence $\zeta^a \zeta^b g_{ab} = g^{-1/2} g_{ab} z^a z^b = z^a z^b \delta_{ab} := z^2$. Thus, the confining string theory derived in this paper is characterized by the parameters,

$$\sigma = 4 \int d^2z [m_1^{-2} F_1(z^2) - m_2^{-2} F_2(z^2)], \quad (\text{G.17})$$

$$\alpha_0^{-1} = -\frac{1}{4} \int d^2z z^2 [m_1^{-4} F_1(z^2) - m_2^{-4} F_2(z^2)], \quad (\text{G.18})$$

$$\kappa = \frac{1}{6} \int d^2z z^2 [m_1^{-4} F_1(z^2) - m_2^{-4} F_2(z^2)]. \quad (\text{G.19})$$

By substituting (G.12) into (G.17), we obtain

$$\sigma = -4J^2 g^2 \rho^{-2} K \frac{\chi}{4\pi^2} \left[\int_{\frac{m_1}{\Lambda}}^\infty 2\pi |z| |d| |z| \frac{K_1(|z|)}{|z|} - \int_{\frac{m_2}{\Lambda}}^\infty 2\pi |z| |d| |z| \frac{K_1(|z|)}{|z|} \right] \quad (\text{G.20})$$

$$= -4J^2 g^2 \rho^{-2} K \frac{\chi}{2\pi} \left[K_0\left(\frac{m_1}{\Lambda}\right) - K_0\left(\frac{m_2}{\Lambda}\right) \right], \quad (\text{G.21})$$

where we have used $K_1(x) = -K_0(x)$, $K_0(\infty) = 0$ and introduced the ultraviolet cutoff Λ ($K_0(0) = \infty$). Incidentally, the asymptotics of $K_p(z)$ for $z > 0$,

$$K_p(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} [1 + O(z^{-1})], \quad (\text{G.22})$$

means that $K_p(z)$ decreases exponentially for large z . After removing the cutoff Λ , the above expression reduces to a finite value,

$$\sigma_{st} \cong \frac{(2Jg)^2}{4\pi} \rho^{-2} K \chi \ln \left(\frac{m_1}{m_2} \right). \quad (\text{G.23})$$

The coefficient of the rigidity term is calculated as

$$\begin{aligned} \alpha_0^{-1} &= J^2 g^2 \rho^{-2} K \frac{\chi}{4\pi^2} \int_0^\infty 2\pi |z| |d| |z| \left[\frac{|z|^2}{m_1^2} \frac{K_1(|z|)}{|z|} - \int_0^\infty |z| |d| |z| \frac{|z|^2}{m_2^2} \frac{K_1(|z|)}{|z|} \right] \\ &= J^2 g^2 \rho^{-2} K \frac{\chi}{4\pi^2} \left[\frac{4\pi}{m_1^2} - \frac{4\pi}{m_2^2} \right]. \\ &= -J^2 g^2 \rho^{-2} K \frac{1}{\pi} < 0, \end{aligned} \quad (\text{G.24})$$

where we have used the integration formula,

$$\int_0^\infty dx x^{\mu-1} K_\nu(ax) = 2^{\mu-2} a^{-\mu} \Gamma\left(\frac{\mu-\nu}{2}\right) \Gamma\left(\frac{\mu+\nu}{2}\right) \quad (\Re\mu > \Re\nu). \quad (\text{G.25})$$

Here note that the integral in (G.24) is finite and we don't have to introduce the cutoff. Finally, the κ is calculated as

$$\kappa = \frac{2}{3} J^2 g^2 \rho^{-2} K \frac{1}{\pi} > 0. \quad (\text{G.26})$$

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